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Application of the Forced Stuart-Landau Model to Cylinder Wake Oscillation

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Abstract

In this article the application of the forced Stuart-Landau equation to describe the wake flow from a circular cylinder in uniform flow under transverse forcing is investigated. Previous work has shown that the forced Stuart-Landau equation predicts multivalued behaviour can occur over a range of forcing frequencies for a sufficiently large forcing amplitude. In practice, this should mean that the wake is hysteretic as a function of the forcing frequency. Both numerical simulations and experiments have failed to find this predicted hysteresis. The resolution appears to be that forced Landau model predicts that, for a particular forcing amplitude, the range of forcing frequencies over which hysteresis is predicted to occur varies in space. This means that the model can predict the wake response (as measured by the oscillation amplitude) almost up to the frequency corresponding to maximum response, but for higher forcing frequencies the predictions deviate from the observations.

Introduction

The complex Stuart-Landau (often just referred to as the Landau) equation has been widely used to model the shedding of vortices in the two-dimensional wake of a cylinder at low Reynolds numbers. Specifically, the different coefficients of the model have been measured from experiments (Sreenivasan *et al.* [11], Provansal *et al.* [8], Schumm *et al.* [9], Albarède & Provansal [1]), and from numerical simulations (Dusek *et al.* [3]). The model has also been applied to other cases such as the transition to the periodic wake for flow past a sphere (Ghidersa & Dusek, [5], and Thompson *et al.* [13]), flow past triangular cross-sectioned cylinders (Zielinska & Wesfreid [15]), and forms the basis of coupled models for interacting bluff body wakes from more than one body. In addition, the *forced* Landau model also appears to have applications for forced periodic bluff body flows. In particular, the experiments of Bishop & Hassan [2] have clearly shown jumps and hysteresis loops in the resonance curves for the amplitude and the phase of the vortex shedding. These resonances appear for particular excitation frequencies. Stansby [10] has shown the existence of resonant horns where the wake is locked to the cross-flow oscillation of the cylinder. These types of behaviours are typical of predicted states of forced damped oscillator systems.

The present study uses numerical simulations to investigate the resonance response of forced cylinder wake flow and compares the results with theoretical predictions from the forced Stuart-Landau model. We focus our attention on the pre-critical regime, where the periodic solution is damped when it is not excited. To our knowledge, the only attempt at modeling the periodically forced wake by a forced Landau equation below the threshold has been by Provansal *et al*. [8]. In this case, the forcing term which is added to the model is a simple harmonic term, having a given amplitude and frequency. Above the threshold, additional third-order terms are involved in the amplitude equation associated with the forced Hopf bifurcation (Walgraef [14]). The solution is then much more intricate with the possibility of the appearance of higher-order resonances and bi-periodic behaviour. A complete mathematical analysis of the different possibilities has been provided by Gambaudo ([4]). In addition, numerical solutions of the forced Stuart-Landau equation in the post-critical regime have been obtained (Olinger [7]).

In this paper, we limit our attention to the pre-critical flow regime where locking is expected (Gambaudo [4]). It was shown by Le Gal *et al.* [6] that, due to the cubic nonlinearity of the Landau equation, the resonance curves can exhibit a hysteresis loop for a certain range of parameters. We study the wake behaviour of a circular cylinder subject to transverse sinusoidal oscillations numerically, to test the predictions from the theory.

Physical Model

The situation under investigation is depicted in Figure 1. A circular cylinder is placed in a uniform flow. If the Reynolds number $Re = UD/\nu$ is above a critical value ($Re_{crit} = 46.4$), the wake forms the characteristic Benard-von Karman vortex street. In this case we are concerned with Reynolds numbers slightly below Re_{crit}. Periodic shedding results only because of low-level transverse forcing; typically, strong shedding results with forcing levels below $U_f/U = 0.1\%$. Here, U_f/U is the transverse velocity amplitude (relative to the inflow velocity) of the forcing.

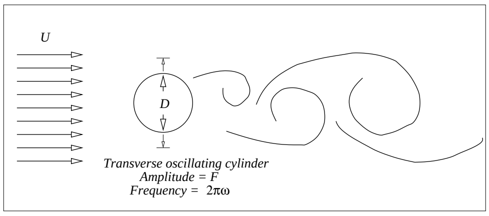

Figure 1: Problem setup showing critical parameters.

Theory

Details of the finer points of the Stuart-Landau model applied to circular cylinder wake transition can be found in Dusek *et al.* [3]. Importantly, the equations can be non-dimensionalised using a physically based-scaling (e.g., [6]), which shows that the critical parameter determining the global behaviour is the ratio of the imaginary to real coefficient of the cubic term. (This is often called the Landau constant and is denoted by c).

In the present analysis, the equations are kept in dimensional form. Only a brief exposition of the theory will be presented here; the reader is referred to the references above for a more detailed and in depth description.

The Stuart-Landau model describes the initial transient linear growth and subsequent saturation of the wake at (pre- and) post-critical Reynolds numbers. The transition of interest here is the Hopf bifurcation of the wake of a circular cylinder, that is, the transition from a steady to a periodic wake as the Reynolds number is increased above a critical value. For this cylindrical body-shape experimental and numerical studies have revealed the transition occurs between $Re = 46$ and 47.

The complex 3rd order Stuart-Landau equation with time-periodic forcing is given by

$$\frac{dA}{dt} = \quad (a_R + i a_I)A - (\ell_R + i\ell_I)|A|^2 A + F e^{i\omega t}, \qquad (1)$$

in which A is a complex-valued function of time t and the parameters a_R, a_I, ℓ_R and ℓ_I are all real. The Landau constant, usually denoted by c, is given by $c = \ell_I/\ell_R$ in this formulation. The equation has been truncated after the cubic term as is the usual case for supercritical transitions since the cubic term is nominally responsible for limiting the initial exponential growth and causing saturation. This is the case for the transition to periodic shedding in the circular cylinder wake. The last term represents the forcing. The forcing amplitude F and frequency ω are taken to be real.

In the absence of forcing (i.e., $F = 0$), Eq. (1) represents the normal form of the Hopf bifurcation which occurs at the critical value of the parameter $a_R = 0$. For $a_R < 0$, the null solution ($A = 0$) is a stable solution of the unforced equation. For a circular cylinder the flow corresponds to steady flow with attached eddies at the rear of the cylinder. For $a_R > 0$, this base state loses its stability and the solution settles down to a time-periodic state (corresponding to Bérnard or Karman vortex shedding). If only the cubic terms are considered, the saturation amplitude is given by $|A| = (a_R/\ell_R)^{1/2}$ and the angular frequency at saturation is given by $a_I - a_R c$ (e.g., [3]). Note that it is known from previous investigations (see [3]) that ℓ_R is positive . The time-scale for the transient approach to this final periodic state is given by a_R^{-1}.

At this stage, the analysis can be simplified by non-dimensionalising the variables to reveal the important governing parameters. This analysis is presented in [6]; only the results will be given here. The analysis shows that if the cubic coefficients are constants, then the system behaviour at saturation is fully determined by the value of the Landau constant, together with the relative forcing level and the frequency at which forcing is applied. The critical value of the Landau constant is $c_{crit} = -\sqrt{3}$. For c less than the critical value, the equations indicate that the saturated state can be multivalued provided the forcing amplitude and frequency are above critical values dependent on c. This is of particular interest because both experimental and numerical determinations of the Landau constant indicate that it is about -3 near bifurcation. In practice, this means that the saturated state is hysteretic—over a specific frequency range, the wake state at saturation corresponding to a particular frequency will be different depending on whether the frequency was approached slowly from above or below.

Equation (1) can be used to predict the *resonance curve*, that is, the variation of the saturated mode amplitude with forcing frequency for a fixed forcing amplitude. This is done by first writing the amplitude in the form

$$A(t) = \rho(t)\exp(i\Phi(t)).$$

By splitting the complex Landau equation into real and imaginary parts and solving for the saturation amplitude, (see [6] for details), the equation governing the saturation amplitude is given by

$$(a_R + \ell_R\rho_{sat}^2)^2 + (a_I + \ell_I\rho_{sat}^2 - \omega)^2 = \frac{F^2}{\rho_{sat}^2}. \qquad (2)$$

This equation can be further manipulated to determine the saturation amplitude ρ_{sat} as a function of ω.

The analysis and main predictions of the model are presented in [6]; we only present a summary here.

From this point it is possible to derive the dependence of the saturated state on the governing parameters including ℓ_R and ℓ_I. (However, whether hysteresis does or does not occur is determined completely by the value of c). The main result is that there exist critical values of ω, $-c$, and F above which the final state is multivalued. It should be possible to observe this behaviour experimentally and numerically.

In order for a multivalued saturated state to occur, parameter c must satisfy $c < -\sqrt{3}$ and ω must satisfy the following relationship

$$(\omega - a_I)/|a_R| > \omega_{crit} \equiv \frac{\sqrt{3}c - 1}{c + \sqrt{3}}. \qquad (3)$$

For any (c, ω) pair which satisfy these conditions, there is a minimum critical forcing frequency above which three final saturated states exist. Two of these flow states are stable and hence are observable in reality, while the other is unstable and generally is not observable. The critical value of the forcing is determined by the following equation

$$F_{crit}^2 = \frac{a_R^3}{\ell_R}\frac{-8}{3\sqrt{3}}\frac{(c^2+1)}{(c+\sqrt{3})^3}. \qquad (4)$$

Expressions for the maximum amplitude and corresponding forcing frequency are given in [6].

Results

The spectral-element method is used to simulate the flow, both below and above the critical Reynolds number, Re_{crit}. The specific implementation is described in Thompson *et al.* [12]. Care has been taken to verify the convergence of the simulations. Resolution and domain size studies indicate the accuracy of the predictions is better than 1%. Figure 2 shows the mesh of macro-elements used for the simulations. Within each element the mesh is further subdivided into 7×7 collocation points. The behaviour of the flow is monitored by recording the velocity components at fixed points in the flow. In particular, the transverse component of the velocity on the centreline is used since it is zero in the pre-transition state.

The saturated wake state at $Re = 48$ is shown in terms of the vorticity field in Figure 3.

Initially the critical Reynolds number was determined using linear extrapolation of the growth rates (a_R) determined at a series of post-critical Reynolds numbers. The critical Reynolds number was evaluated as $Re_{crit} = 46.4$ in agreement with previous numerical predictions for this flow.

The Landau model parameters for $Re = 46$, were estimated as $a_R = -0.00088$, $a_I = 0.3708$. This allows an estimation of the critical forcing frequency from equation (3) as $\omega_{crit} = a_I - c|a_R|$. Given $c = -3.2$ at $Re = 46$ based on the determination of the frequency shift of the saturated states at higher Re, this gives the critical forcing frequency of $\omega_{crit} = 0.374$ only about 1% in excess of the natural oscillation frequency. In addition, the critical forcing amplitude can be calculated to

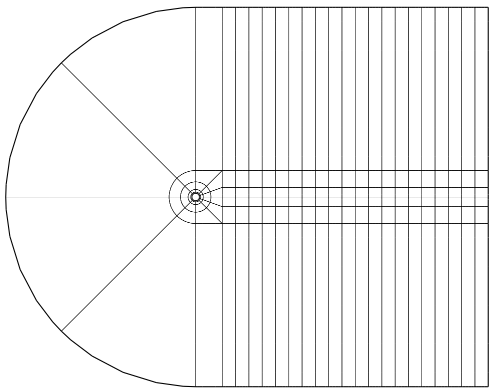

Figure 2: The spectral-element mesh used for the simulations.

Figure 3: Vorticity field in the wake at $Re = 48$.

correspond to a perturbation velocity amplitude $U_f/U < 0.1\%$. Thus very low amplitude forcing should be sufficient to induce hysteretic behaviour.

To determine the cubic model coefficients, the unforced flow was simulated at $Re = 46$ for many hundreds of shedding cycles with the velocity field initialised from a solution at $Re = 200$ where shedding is extremely strong. After about 50 shedding periods, the effect of the initial state convects out of the system, and the evolution of the wake towards the steady state is presumed to be determined by the unforced Landau model. This allows the determination of the parameters ℓ_R and ℓ_I in a similar way to the method used by Dusek *et al.* [3]. These parameters were also estimated by extrapolating values determined at post-critical Reynolds numbers. Both methods provide similar estimates. In fact, ℓ_R varies with downstream position (as has been found by others (e.g., see [3])), but more surprising the Landau constant $c = \ell_R/\ell_I$ also varies significantly with distance. Since the Landau constant determines the frequency shift of the saturated state at post-critical Reynolds numbers, and the final flow state is locked at all points in space, this value should be independent of position. Most likely, the values determined here as a function of position vary in space because the wake shape varies considerably as it evolves, so that the measurement of the transverse velocity at a fixed point is not necessarily an ideal measure of the *global* mode amplitude (as required by the Landau model). Despite this, the transverse velocity has been used successfully in the past (e.g., Dusek *et al.* [3]), to model this transition. In practice, it means that higher-order terms may need to be included to accurately describe transition if a point velocity is used to monitor mode amplitude.

A series of simulations of the forced flow were then performed for $Re = 46$. Two types of forcing were applied resulting in essentially equivalent results. In the first case, the cylinder was oscillated in the transverse direction. In the second case, a transverse oscillation was added to the flow at inlet and side boundaries. Although there were slight quantitative differences in the resonance curves, there were no qualitative differences. The results reported here are for the forcing applied at the external boundaries. For each forcing amplitude investigated, the wake was evolved until it reached a periodic state for a set of forcing frequencies. This allowed the *resonance* curves to be constructed as shown in Figure 4. The three curves shown correspond to different forcing amplitudes of $U_f/U = 0.1, 0.05$ and 0.025%. The amplitude was measured at a point 7D downstream of the centre of the cylinder. Overlaid are the resonance curves predicted from the forced Stuart-Landau model for comparison. Clearly, the actual resonance curves are well predicted for lower frequencies but are not well-predicted near the predicted hysteresis ranges, although the curves are bent towards the right side. Also the maximum of resonance amplitude is lower than that predicted by the model. These results are qualitatively duplicated at other downstream locations.

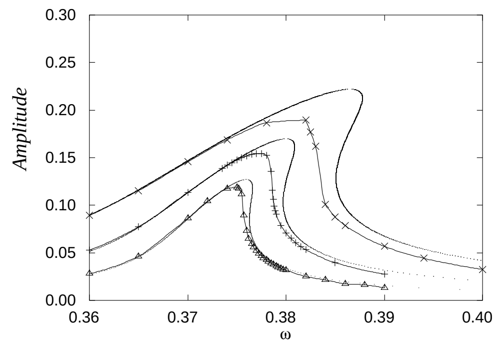

Figure 4: Predicted resonance curves. The symbols denote resonance points determined by the numerical simulations. From lowest to highest the curves correspond to forcing levels of $U_f/U = 0.025, 0.05$ and 0.1%. Also shown are the corresponding predicted resonance curves from the forced Landau model. The amplitude corresponds to the transverse velocity component at 7D downstream of the cylinder.

Thus, although the Stuart-Landau model predicts a multivalued wake state, there is no indication that this occurs in the real system. Note that experimental results (Le Gal *et al.* [6]) concur with this finding. However, the model does appear to predict accurately at least part of the resonance curves, especially at lower frequencies.

A possible explanation for this breakdown in the model may be understood through reference to Figure 5. This shows the predicted resonance curves (from the Landau model) as a function of downstream distance for the same forcing level. The variation of the cubic coefficients of the model mean that the range of frequencies for which the wake is multivalued, varies considerably with downstream position. Describing the two wake states as *low* amplitude and *high* amplitude, it can be seen that at low forcing frequencies all points in the wake prefer the high amplitude wake state. In particular, this applies to points to the left of the dotted line on the figure. Similarly, for (very) high frequencies the low wake state is preferred. However, in between, different points in the wake prefer different wake states depending on position. The predicted resonance curve corresponding to $x = 7D$ downstream is marked on the diagram. This is the same curve shown in Figure 4 for the intermediate forcing amplitude. The range over which hysteresis can occur is different from points upstream and downstream. In particular, considering the case where the forcing frequency is slowly increased, the wake state should remain in the *high* state until the point corresponding to the maximum resonance amplitude is approached. However, near that frequency, at points slightly downstream, the wake can only exist in the *low* state. This provides a hypothetical explanation for why the measured maximum amplitude at resonance is less than the predicted amplitude from the model. It also explains why the wake fails to show the multivalued behaviour predicted because for any point in the wake at a forcing frequency corresponding to the multivalued range, there are other points where the wake prefers to be in only either the low or high amplitude state. In effect, the hysteresis range is smoothed out by the interaction (of the oscillators) at different points in the wake.

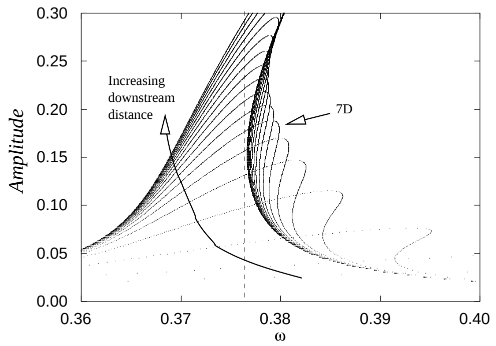

Figure 5: Predicted resonance curves from the forced Stuart-Landau model as a function of downstream distance using measured local values of the parameters in the Landau equation.

Conclusions

The forced Stuart-Landau equation is applied to modelling the behaviour of the wake from a circular cylinder close to the transition Reynolds number where periodic shedding first occurs. While the model applied pointwise predicts hysteresis as a function of forcing amplitude, numerical simulations and experiments show this not to be the case. However, the model does accurately predict the shape of the resonance curves at low (and high) forcing frequencies away from the hysteretic range. It seems likely that the wake could be described by a set of interacting forced Landau model systems with the interaction terms limiting the possibility of hysteresis, at least in this case. This possibility is currently being pursued.

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