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VORTEX MODELS FOR FEEDBACK STABILIZATION OF BLUFF BODY WAKE FLOWS

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1. Introduction

The goal of this presentation is to review recent progress concerning the design of feedback control strategies for bluff body wake flows based on point vortices. This investigation grows out of a long–term research effort which seeks to integrate rigorous methods of modern control theory and computational fluid dynamics. We will use a combination of mathematical analysis and numerical computation to study properties of a family of flow control algorithms and will focus on circular cylinder wake flows which are canonical examples of massively separated flows. In principle, application of the linear control theory to systems described by partial differential equations (PDEs) is relatively well understood, however, in practice even the design of "simple" linear control strategies, such as the Linear Quadratic Regulator (LQR), may result in computationally intractable problems when applied to discretizations of the full Navier–Stokes system [1]. Therefore, in order to facilitate synthesis and application of such control strategies, it is necessary to introduce reduced–order models of the Navier–Stokes system and in this investigation we study one such family of reduced–order models.

2. The Föppl System as Reduced–Order Model

In this research we are interested in stabilizing the steady symmetric flow past a circular cylinder which is known to become unstable for $Re \gtrsim 46$. In order to simplify the mathematical description, we will assume that the system satisfies the steady–state Euler equations which can be written in the form

$$\begin{cases} \Delta\psi = f(\psi) & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \\ \psi \to U_\infty y & \text{for } |(x,y)| \to \infty, \end{cases} \tag{1}$$

where Ψ is the streamfunction and the right–hand side function f is a priori undetermined. Taking this function in the form $f(\Psi) = -\omega H(\Psi - \Psi_0)$, where $H(\cdot)$ is the Heaviside function, we obtain a family of Prandtl–Batchelor flows [3], characterized by constant–vorticity vortex patches embedded in irrotational flow, as solutions of problem (1). Assuming that the circulation of every vortex region is fixed results in a one–parameter family of solutions of (1) depending on the area of the vortex region [3] (see Fig. 1a). Taking the limit of the vanishing vortex area reduces the Prandtl–Bachelor flow family to an equilibrium point vortex system discovered by Föppl [2] in 1913 (Fig. 1b). Analysis of the linear stability of the Föppl equilibrium shows that it is unstable and, in addition to a linearly growing mode associated with a real positive eigenvalue, is also characterized by a decaying mode associated with a real negative eigenvalue and a neutrally stable oscillatory mode associated with a conjugate pair of purely imaginary eigenvalues. These stability properties make the Föppl system a feasible candidate for a reduced–order model of the onset of the vortex shedding instability in bluff body wakes.

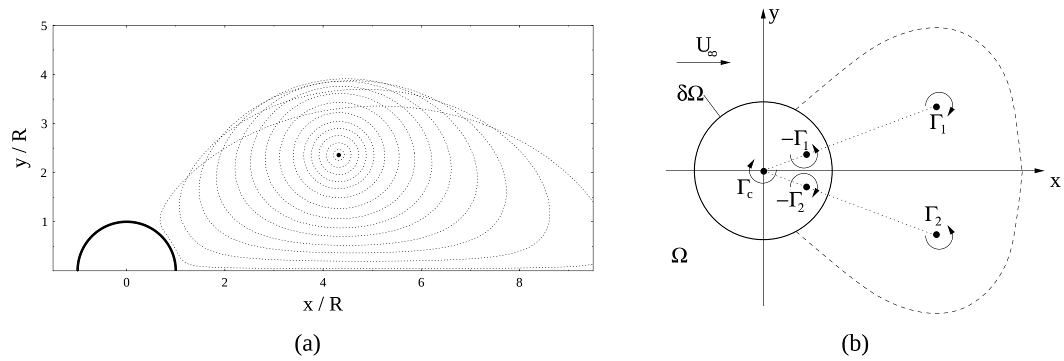


Figure 1. (a) Boundaries of the vortex patches with different areas and constant circulation obtained as solutions of (1) and the limiting point vortex Föppl system (represented by a solid circle), (b) schematic showing the location of the singularities in the Föppl system with control representing the cylinder rotation.

3. Control Design

Our goal is to stabilize the steady symmetric wake flow represented, for the control design purposes, by the unstable equilibrium of the Föppl system as a reduced–order model. The flow actuation (system input) has the form of the cylinder rotation and is represented in the Föppl system as a vortex with the circulation $\Gamma_C = \Gamma_C(t)$ located inside the obstacle, whereas the system output has the form of velocity measurements $\mathbf{y}$ at the flow centerline. Using $\mathbf{x} \in \mathbb{R}^4$ to denote the perturbation variables (i.e., perturbations of the vortex positions around the equilibrium), the linearization of the Föppl system around this equilibrium can be expressed in the canonical state–space representation as [4]

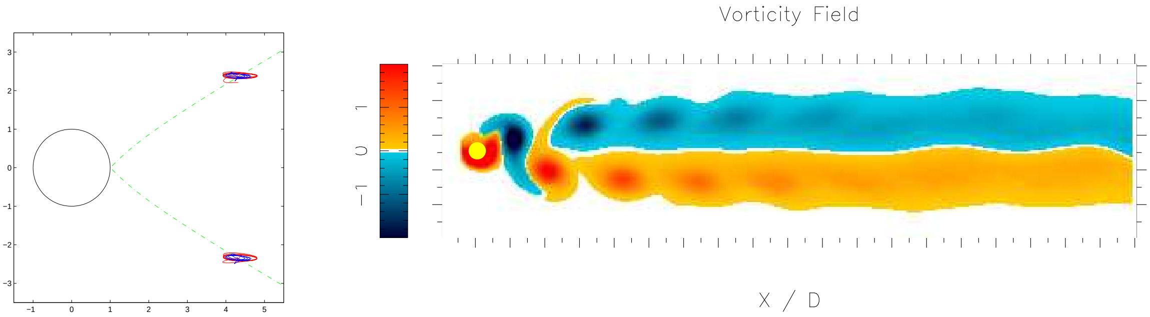
$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\Gamma_C, \tag{2a}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\Gamma_C, \tag{2b}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are suitable matrices. We seek to determine the control in the *feedback* form $\Gamma_C(t) = -\mathbf{K}\mathbf{x}(t)$, so that it will stabilize model equation (2a) and at the same time will minimize the cost functional $\mathcal{J}(\Gamma_C) = \frac{1}{2}\int_0^\infty (\mathbf{y}^T\mathbf{Q}\mathbf{y} + \Gamma_C R \Gamma_C)\,dt$, where $R > 0$ and $\mathbf{Q}$ is a suitably chosen positive–definite weighing matrix. Before we can devise a control algorithm, we need to verify that model system (2) has an appropriate internal structure. It was shown in [4] that problem (2) is fully *observable*, however, it is not *controllable*. Performing the Kalman decomposition in order to transform system (2) to the minimal representation, i.e., one which is both observable and controllable, shows that the neutrally stable oscillatory modes are in fact not controllable, so the whole system remains *stabilizable*. The stabilization problem is solved by constructing a linear–quadratic–Gaussian (LQG) compensator [4] and in Fig. 2a we show the results concerning LQG stabilization of the Föppl equilibrium. We note that the vortex trajectory is indeed stabilized, however, instead of returning to the equilibrium, the trajectory lands on a circular orbit circumscribing the equilibrium. The same LQG approach was then applied to stabilization of the circular cylinder wake at $Re = 75$ (Fig. 2b). We observe that, while the far wake is remarkably symmetrized, the level of oscillations in the near wake region is in fact increased. Properties of the Föppl system responsible for the behavior observed in these two cases are investigated next.

4. Center Manifold Analysis

It is well–known that, if a linearization of a nonlinear system possesses pairs of purely imaginary eigenvalues, then such linearization may not provide conclusive information about stability of the



(a) (b)

Figure 2. (a) Trajectories of the Föppl vortices with the LQG control, (b) instantaneous vorticity field in a cylinder wake at $Re = 75$ with the LQG control.

original nonlinear system and higher–order information must be analyzed. To this end we consider the minimal representation of system (2) with the feedback control $\Gamma_C = -\mathbf{Kx}$

$$\frac{d}{dt}\begin{bmatrix}\boldsymbol{\xi}\\ \boldsymbol{\eta}\end{bmatrix} = \begin{bmatrix}\mathbf{A}_{11} & 0\\ 0 & \mathbf{A}_{22}-\mathbf{BK}\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}\\ \boldsymbol{\eta}\end{bmatrix} + \begin{bmatrix}\mathbf{g}_1(\boldsymbol{\xi},\boldsymbol{\eta})\\ \mathbf{g}_2(\boldsymbol{\xi},\boldsymbol{\eta})\end{bmatrix}, \tag{3}$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ represent, respectively, the controllable and uncontrollable parts of the state of the Föppl system with the feedback control and the matrix $\mathbf{A}_{11}$ has purely imaginary eigenvalues only. In [5] we proved the following two theorems in regard to system (3):

Theorem 1. *System* (3) *possesses an invariant (center) manifold given by the function* $\boldsymbol{\eta} = \boldsymbol{\phi}(\boldsymbol{\xi}) = \mathbf{0}$.

Theorem 2. *For sufficiently small initial data the reduced system*

$$\frac{d}{dt}\boldsymbol{\xi} = \mathbf{A}_{11}\boldsymbol{\xi} + \mathbf{g}_1(\boldsymbol{\xi},\mathbf{0}), \tag{4}$$

obtained via an invariant reduction of system (3)*, possesses stable periodic orbits.*

The significance of these results concerning the observed behavior of the Föppl system under feedback control is as follows. Theorem 1 implies that the controllable and uncontrollable parts of the state are essentially uncoupled. Therefore, as soon as the control stabilizes the unstable mode, the system trajectory converges to the center manifold $\boldsymbol{\xi} = \mathbf{0}$. Since this manifold is in fact spanned by the uncontrollable modes, the dynamics on this manifold is unaffected by the flow actuation and, as asserted by Theorem 2, stable periodic oscillations are observed. We conclude that the presence of this center manifold is clearly an undesirable effect from the control point of view. Next we attempt to modify the internal structure of the Föppl system so as to disrupt the center manifold.

5. Beyond the Classical Föppl System

In Section 2. we argued that the classical Föppl system represents an extreme member of the Prandtl–Batchelor family of vortex flows. In [6] we showed that it is in fact possible to construct point vortex systems corresponding to the Prandtl–Batchelor flows with finite area vortex patches. This can be accomplished by adding higher–order terms representing corrections due to the finite size of the vortex patch to the classical Föppl system. As shown in [6], the equilibria of such higher–order Föppl systems form loci parametrized by the area of the vortex patch and the truncation order (Fig. 3a). In addition to a range of properties interesting from the mathematical point

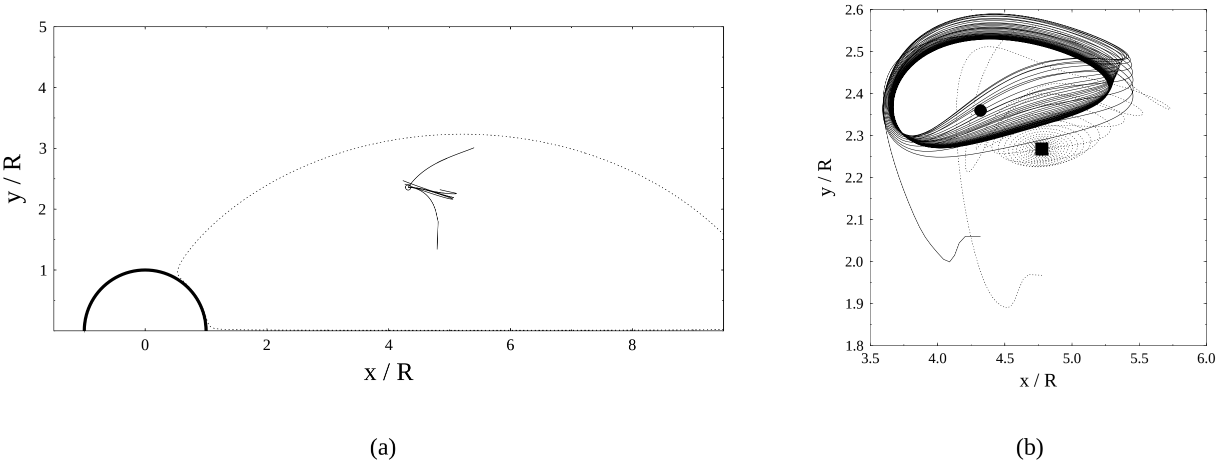


Figure 3. (a) Loci of the higher–order equilibria parametrized by the area of the vortex region in the Prandtl–Batchelor solution for different truncation orders (the dotted line represents the boundary of a vortex region, whereas the thick solid line represents the obstacle), (b) trajectories of the state of (solid line) the classical and (dotted line) higher–order Föppl system stabilized with an LQG compensator in the neighborhood of the corresponding equilibrium solutions.

of view, such higher–order Föppl systems have an important characteristic relevant for our control applications, namely, the uncontrollable modes are now exponentially, rather than just neutrally, stable, This means that a center manifold is no longer present in this new reduced–order model and, as shown in Fig. 3b, the LQG compensator is now able to completely stabilize the equilibrium. Control–theoretic advantages of the higher–order Föppl systems as reduced–order models are being now investigated. It is anticipated that controllers designed based on such higher–order systems will be characterized by more robust performance, especially when applied to actual systems.

Acknowledgments

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