On Frictionally Constrained Cracks

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Summary

The classical approaches for modelling the mechanics of cracks is most likely to be satisfied in situations where the dominant states of external loading are tensile and results in crack opening. A consequence of crack closure is the development of frictional constraints at the crack surfaces. Frictionally constrained cracks can be conveniently handled through boundary element techniques. This paper examines the mechanics of frictionally constrained regions that contain wing-cracks at their extremities. The stress intensity factors at the tips of these wing cracks are influential in determining the path of crack extension during compressive loading of cracks that are obliquely oriented to the direction of compressive loading.

Introduction

The modelling of open plane cracks in elastic solids dates back to the classical work of Griffith [1], and the subject area is now an integral part of modern fracture mechanics and to materials engineering (see e.g. [2-4]). In conventional treatments, it is generally assumed that the crack surfaces are maintained in an open condition even during the application of predominantly compressive forms of external loading. This is a limitation of the theoretical treatment and it is not uncommon to find computations of Mode I stress intensity factors for situations where the crack tip does in fact experience closure. Examples of analyses of cracks with closed tips are available in the literature, but these are largely in relation to the study of crack tips at bi-material interfaces and not in the context of cracks in homogeneous media. Mechanics of cracks with closed tips has been investigated mainly in connection with the evaluation of brittle fracture in geomaterials such as rocks and concrete, where the primary mode of loading is associated with compressive stress states [5]. In a material that contains a distribution of cracks with random orientations, it is likely that some cracks will experience opening and others will clearly experience closure. There is the further possibility that, depending on their orientation, certain cracks could experience frictional closure. The theoretical modelling of a random assemblage of micro-cracks that can experience all three modes of responses is a non-routine exercise. This aspect is perhaps better addressed through consideration of a phenomenological micromechanical damage model with stress state-dependent mechanical responses, involving both elastic and non-linear phenomena. The objective of the present paper is to assess the role of frictional constraints that can exist at an interface on the mechanics of a wing cracks that can be present at the edges of the constrained

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interface. The region is first subjected to an isotropic bi-axial stress state with the frictional interface constraint relaxed and in this stressed state it is subjected to an incremental uniaxial compressive stress state that maintains the frictional contact and induces opening of the wing crack. The analysis of the mechanics of wing cracks is particularly important to establishing the mode of quasi-static crack extension during compressive loading of brittle geomaterials [5]. In this sense, the tensile splitting of brittle geomaterials during compressive loading of cylindrical specimens is assumed to a consequence of the extension of wing cracks.

The modelling of the mechanics of wing cracks that occur at the extremities of frictionally constrained regions can be approached via either finite element or boundary element techniques. In this study, the boundary element method is chosen in view of certain inherent advantages. Firstly, the domain containing the plane crack and the constrained region are elastic except for the non-linear constraints that are imposed on the frictionally constrained regions. Secondly, the nonlinear processes that can result from friction, slip, dilatancy, interface plasticity, etc., are restricted to only the known contact surfaces and the techniques for examining the various regions over which the specific constraints apply are reduced to the determination of the extent of a region within this contact surface. Thirdly, the boundary element schemes, when combined with special singular elements, can be used to determine quite accurately the stress intensity factors at the tips of the wing cracks. Finally, the boundary element scheme can be used with the minimum of mesh adaptation to track the path of crack during extension of wing cracks. The purpose of this paper, however, is mainly to examine the influence of frictional constraints at an interface on the stress intensity factors at the tip of a wing crack emanating from the boundaries of the frictionally constrained region.

Incremental Formulations

The boundary element approach for the modelling of elastic continua is relatively well established [6, 7]. An incremental formulation is necessary for the modelling of interface non-linearity encountered at the crack surfaces. The incremental form of the boundary integral equation for a region $V^{(\rho)}$ with surface $\Gamma^{(\rho)}$ can be written in the form

$$c_{ij}\,\dot{u}_j^{(\rho)} + \int_{\Gamma^{(\rho)}} P_{ij}^{*(\rho)} \dot{u}_j^{(\rho)}\, d\Gamma = \int_{\Gamma^{(\rho)}} u_{ij}^{*(\rho)} \dot{P}_j^{(\rho)}\, d\Gamma \tag{1}$$

where $i, j = 1,2,3$ (or x, y, z); $\dot{u}_j^{(\rho)}$ and $\dot{P}_j^{(\rho)}$ are, respectively, the incremental values of the boundary displacements and boundary tractions and expressions for $u_{ij}^{*(\rho)}$ and $P_{ij}^{*(\rho)}$ are given in [6]. The regions in contact can be subjected to the conventional displacement and traction boundary conditions as well as interface conditions. On an interface region with non-linear constraints we have

$$\dot{P}_i = \dot{R}_i + K^*_{ij}\dot{u}_j \tag{2}$$

where $\dot{R}_i$ are increments of a residual traction and K^*_{ij} are stiffness coefficients derived through considerations of the non-linear constraints. Upon boundary element discretization of the domain, the integral equation can be converted to its matrix equivalent, which can be written in the form

$$[\mathbf{H}]\{\dot{\mathbf{u}}\} = [\mathbf{G}]\{\dot{\mathbf{P}}\} \tag{3}$$

where $[\mathbf{H}]$ and $[\mathbf{G}]$ are the boundary element influence coefficients matrices and $\{\dot{\mathbf{u}}\}$ and $\{\dot{\mathbf{P}}\}$ are the incremental displacement and traction vectors and sub-sets of which together will form the appropriate set of unknowns. If the configuration of the boundary and the interface conditions are defined at any level of deformation, we can obtain a final system equation in the form

$$[\mathbf{A}]\{\dot{\mathbf{U}}\} = \{\dot{\mathbf{B}}\} \tag{4}$$

from which, either the boundary or the interface unknowns can be determined. Further details of the incremental procedures are given in [8,9].

Interface Responses

In this paper, we shall briefly outline a treatment of the interface response, which can be used to model Coulomb friction at contact zones. In view of the non-linear nature of the interface response, it is necessary to adopt an incremental approach to the formulation of the constitutive responses. The interface is also regarded as a distinct two-dimensional surface that is void of a dimension normal to the plane, with the result, the interface responses must be formulated in relation to the incremental relative displacements $\dot{\Delta}_i$ that take place at the Euclidean contacting plane. We assume that these incremental relative displacements consist of an elastic or recoverable component $\dot{\Delta}_i^{(e)}$ and an irrecoverable or plastic component $\dot{\Delta}_i^{(p)}$: i.e. $\dot{\Delta}_i = \dot{\Delta}_i^{(e)} + \dot{\Delta}_i^{(p)}$, where for an interface the subscripts i (or j) can be assigned notations applicable to the local interface coordinates. For purposes of the presentation, we shall denote the values applicable to i and j applicable to an interface by x, y, z with the assumption that the direction z corresponds to the normal to the Euclidean plane at a point on the interface. The elastic component of the incremental displacement $\dot{\Delta}_i^{(e)}$ is related to the component of the corresponding increment of traction $\dot{t}_i$ through the linear constitutive response, $\dot{t}_i = \tilde{k}_{ij}\dot{\Delta}_j^{(e)}$, where $\tilde{k}_{ij}$ are the linear stiffness coefficients of the interface and summation over the repeated indices is implied. This linear elastic response will persist as long as the tractions acting on the interface will not induce failure at the interface. To assess this limiting condition it is necessary to postulate a failure criterion for the interface. From developments in the theory of plasticity, [10], the incremental elastic-plastic constitutive response for the interface can be written as

$$\dot{t}_i = \left[\tilde{k}_{ij} - \frac{1}{\psi} \frac{\partial \Phi}{\partial t_l} \tilde{k}_{il} \tilde{k}_{mj} \frac{\partial F}{\partial t_m} \right] \dot{\Delta}_j = \tilde{k}_{ij}^{(ep)} \dot{\Delta}_j \tag{5}$$

If the failure criterion $F(t_i)$ and the plastic potential $\Phi(t_i)$ for the interface are known, then we can define the elastic-plastic stiffness $\tilde{k}_{ij}^{(ep)}$. For an interface with Coulomb frictional phenomena and for the special case where $\tilde{k}_{xx} = \tilde{k}_{yy} = \tilde{k}_s$; $\tilde{k}_{zz} = \tilde{k}_n$ with all other $\tilde{k}_{ij} \equiv 0$, the non-symmetric elastic-plastic stiffness matrix is given by

$$\tilde{k}_{ij}^{(ep)} = \frac{1}{(t_x^2 + t_y^2)} \begin{bmatrix} \tilde{k}_s t_y^2 & -\tilde{k}_s t_x t_y & -\mu \tilde{k}_n t_x \sqrt{(t_x^2 + t_y^2)} \\ -\tilde{k}_s t_x t_y & \tilde{k}_s t_x^2 & -\mu \tilde{k}_n t_y \sqrt{(t_x^2 + t_y^2)} \\ 0 & 0 & -\tilde{k}_n (t_x^2 + t_y^2) \end{bmatrix} \tag{6}$$

A Wing Crack Problem

The application of boundary element methods to the elastostatic analysis of planar crack problems is relatively well established. Extensive accounts of both fundamental results and applications of boundary element techniques are given by Cruse and Wilson [11] and recent developments are given in [4] and [12, 13]. Since the exact stress singularity at the tip of a planar crack can be incorporated within the boundary element scheme through the *singular traction quarter-point boundary element*, this enables the calculation of stress intensity factors for various modes of deformation of the crack tip. We consider the problem of skew wing cracks that emanate from the extremities of a frictionally constrained region of length $2c$ (see inset sketch in Figure 1). The frictionally constrained region is inclined at 60^0 to the vertical axis. Two skew symmetrically placed wing cracks extend from the ends of the frictionally constrained region. For the purposes of the computations, these are oriented at 15^0 to the plane of the frictionally constrained region. The length of a wing crack l is taken as $0.25c$. The region containing the frictionally constrained wing cracks is first subjected to an *isotropic compressive stress field* σ_0, with the frictional constraints in the interface region rendered inactive. With σ_0 held constant, the crack is then subjected to a compressive incremental stress $\dot{\sigma}$, and the crack opening (Mode I) stress intensity factor at the tip of the wing crack is evaluated. The parameters required for the computational modelling include the normalized magnitude of the isotropic stress (σ_0 / G); the coefficient of friction μ and the relative magnitudes of the non-dimensionalized shear and normal stiffnesses k_s / G and k_n / G. Figure 1 illustrates the variation of the Mode I stress intensity factor at the crack tip for various values of the normalized incremental stress $\dot{\sigma} / \sigma_0$ and as a function of the Coulomb friction μ in the constrained region. The computational results exhibit a consistent variation with μ, with a friction coefficient of $\mu > 1$ corresponding roughly to a fully bonded constraint. The complete frictional constraint essentially reduces the analysis to two obliquely oriented independent cracks of length l that are separated by a finite distance. This idealization is applicable to situations where the frictionally constrained interface does not exhibit either frictional slip or interface dilatancy.

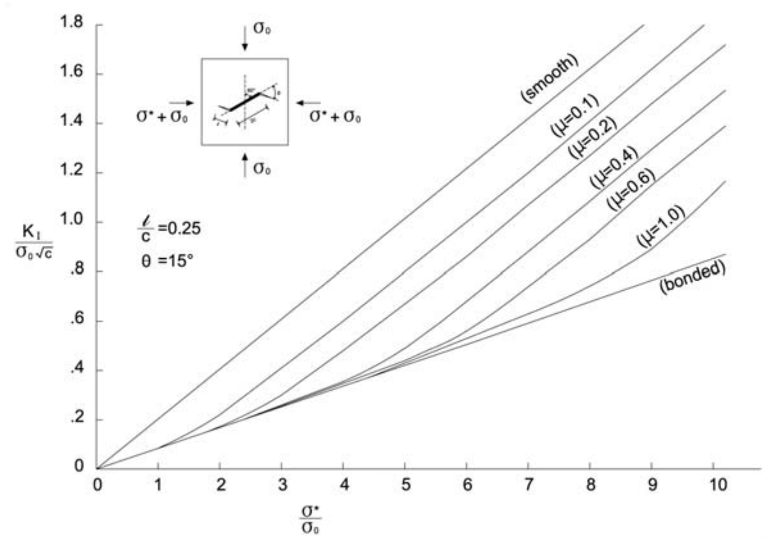

Figure 1. The influence of interface friction on the Mode I stress intensity factor at the boundary of a wing crack

Concluding Remarks

The paper presents the application of the incremental boundary element technique to the study of the two-dimensional problem involving a frictionally constrained region, the extremities of which contain skew-symmetrically placed wing cracks. The Coulomb frictional constraint is applied only when the entire region is first subjected to an isotropic biaxial compressive stress state. The Coulomb friction can be altered to recover the limiting cases of frictionless slip and frictional locking in the constrained region. The results of additional computations indicates that in general, an increase in the friction has the effect of reducing the magnitude of the Mode I stress intensity factor at the tips of the wing cracks, for a wide range of their orientations.

References

1 Griffith, A.A. (1922) The phenomena of rupture and flow in solids, *Phil. Trans. Roy. Soc*., **A221**, 163-198.

2 Sih, G.C. (1991) *Mechanics of Fracture Initiation and Propagation*, Kluwer Academic Publishers., Dordrecht, The Netherlands.

3 Murakami, Y. (1987) *Stress Intensity Factors Handbook, Vols. 1 and 2*, Pergamon Press, Oxford.

4 Aliabadi, M.H. (1997) Boundary element formulations in fracture mechanics, *Appl. Mech. Rev*., **50**: 83-96.

5 Jaeger, J.C and Cook, N.G. W.(1976) *Fundamentals of Rock Mechanics*, Chapman and Hall, London.

6 Brebbia, C.A., Telles, J.C.P. and Wrobel, L.C. (1984) *Boundary Element Techniques. Theory and Applications in Engineering*, Springer-Verlag, Berlin.

7 Gaul, L., Kögl, M. and Wagner, M. (2003) *Boundary Element Methods for Engineers and Scientists. An Introductory Course with Advanced Topics*, Springer-Verlag, Berlin.

8 Selvadurai, A.P.S. and M.J. Boulon, (Eds. (1995) *Mechanics of Geomaterial Interfaces*, Elsevier Sci. Publ. Co., The Netherlands.

9 Selvadurai, A.P.S. (2000) On incremental boundary element procedures for frictionally constrained interfaces, *Proc. IUTAM/IACM/IABEM Symp. Adv. Math. And Comp. Mech. Aspects of the Bound. Elem. Meth.*, (T. Burczyinski, Ed.) Kluwer Acad. Publ., 339-349.

10 Davis, R.O. and Selvadurai, A.P.S. (2002) *Plasticity and Geomechanics*, Cambridge: Cambridge University Press

11 Cruse, T. and Wilson, R.B. (1977) *Boundary Integral Equation Methods for Elastic Fracture Mechanics*, AFOSR-TR-0355.

12 Selvadurai, A.P.S. and ten Busschen, A. (1995) Mechanics of the segmentation of an embedded fiber. Part II, Computational modelling and comparison, *J. Appl. Mech.,* **62**: 98-107.

13 Shiah, Y.C and Tan, C.L. (2000) Fracture mechanics analysis in 2D anisotropic thermoelasticity using BEM, *Comp. Mech. Engng. Sci*., **1**: 91-99.