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# Shakedown Design of Frames with Constraints on Deformations

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## Summary

The minimum volume design of elastic plastic frames subjected to a combination of fixed and cyclic loads is studied. The optimal design problem is formulated in such a way that the structure is able to elastically adapt to the loads in serviceability conditions and to plastically adapt to very high intensity loads. Furthermore, suitable limits on the plastic deformations occurring at the limit state of the plastic shakedown are given as well as a chosen measure of the plastic deformation related to the elastic shakedown is suitably bounded. A suitable solution procedure is utilized.


## Introduction

The continuous progress in technology, the upgrading regarding the computational techniques and the related software incites the designers to require more and more efficiency to the structures. Actually, in the last decades the researchers addressed many efforts to structural optimization, providing many original formulations of the optimal design as well as several interesting contributions related to the computational procedures. The different formulations of the search problem substantially depend on the special limiting criterion imposed on the structure behaviour. So, the elastic optimal design, the elastic shakedown optimal design, the plastic shakedown optimal design, the standard limit design and the multicriteria optimal design (see, e.g., [1-5]) have been developed. Whatever the special formulation is utilized, it is very useful to know if the optimal structure, at the prescribed limit state, fulfils special limits on its ductility behaviour. Actually, in the above referred formulations limits on the structure ductility have been disregarded. Some contributions on this topic have been proposed for elastic shakedown design as well as for standard limit design (see, e.g., [6,7]). The present paper is devoted to propose a formulation of the optimal design of elastic perfectly plastic frames subjected to a combination of fixed and cyclic loads, simultaneously according to a plastic shakedown criterion and to an elastic shakedown one, each related to suitably chosen load multipliers, and with constraints on the plastic strains related to the limit state of plastic shakedown as well as on some suitably chosen measure of the plastic deformations related to the elastic shakedown.

## Structural model and steady-state behaviour

Let us consider a frame constituted by n elastic perfectly plastic elements subjected to quasi-statically variable loads within the interval $\left(0 \le t \le t_f\right)$, being t a monotonically

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increasing variable aimed at specifying the loading sequence. Let us assume that each beam element is constituted by one or more purely elastic beam portions at the ends of which the rigid plastic hinges are located. In the hypothesis of small displacements and strains, the following relationships hold, $\forall\, t \in (0, t_f)$:

$$\mathbf{v}_1 = \mathbf{C}_1\mathbf{u} \tag{1a}$$

$$\mathbf{v}_2 = \mathbf{C}_2\mathbf{u} \tag{1b}$$

$$\mathbf{q}_2 = \mathbf{v}_2 - \mathbf{H}\mathbf{v}_1 \tag{1c}$$

$$\mathbf{Q}_2 = \mathbf{D}^{e}\left(\mathbf{q}_2 - \mathbf{q}_2^{*} - \mathbf{q}_2^{p}\right) \tag{2}$$

$$\mathbf{Q}_1 + \widetilde{\mathbf{H}}\mathbf{Q}_2 + \widetilde{\mathbf{H}}^{*}\mathbf{f}^{*} = \mathbf{0} \tag{3a}$$

$$\widetilde{\mathbf{C}}_1\mathbf{Q}_1 + \widetilde{\mathbf{C}}_2\mathbf{Q}_2 = \hat{\mathbf{F}} \tag{3b}$$

$$\mathbf{q}_2^{p} = \mathbf{H}^{p}\mathbf{p} \tag{4}$$

$$\mathbf{P} = \widetilde{\mathbf{H}}^{p}\mathbf{Q}_2 + \widetilde{\mathbf{H}}^{p*}\mathbf{f}^{*} \tag{5}$$

$$\varphi(\mathbf{P}) \equiv \widetilde{\mathbf{N}}\mathbf{P} - \mathbf{R} \le \mathbf{0}\,,\ \dot{\lambda} \ge \mathbf{0}\,,\ \widetilde{\varphi}\dot{\lambda} = 0\,,\ \widetilde{\dot{\varphi}}\dot{\lambda} = 0 \tag{6}$$

$$\mathbf{p} = \int_0^t \mathbf{N}\dot{\lambda}(\tau)\,d\tau \tag{7}$$

with $\mathbf{v}_1$ and $\mathbf{v}_2$ boundary cross section displacement vectors of the beam elements, $\mathbf{C}_1$ and $\mathbf{C}_2$ compatibility matrices, $\mathbf{u}$ structure node displacement vector, $\mathbf{H}$ compatibility matrix related with the rigid motion of the elements, $\mathbf{q}_2$ total deformation, $\mathbf{D}^{e} = \left(\Phi^{e}\right)^{-1}$ internal elastic stiffness matrix of the elements constrained against any rigid motion by clamping the initial cross sections, $\mathbf{Q}_2$ generalized stress vector evaluated at the element free cross sections, $\mathbf{q}_2^{*}$ and $\mathbf{q}_2^{p}$ deformation vectors related to the actions present along the beams and to the plastic strains $\mathbf{p}$ evaluated at the plastic hinges, respectively, $\mathbf{Q}_1$ generalized stress vector evaluated at the element initial cross sections, $\mathbf{f}^{*}$ external loads applied on the beams, $\hat{\mathbf{F}}$ external loads applied on the structure nodes, $\mathbf{H}^{p}$ and $\mathbf{H}^{p*}$ compatibility matrices analogous to $\mathbf{H}$, $\mathbf{P}$ generalized stress vector evaluated at the plastic hinge sections, $\varphi$ yield function vector, $\mathbf{N}$ matrix of the unit external normals to the yield surface, $\mathbf{R}$ plastic resistance vector and $\dot{\lambda}$ plastic multiplier vector.

By means of the usual mathematics, equation set (1)-(7) can be rewritten as follows:

$$\mathbf{K}\mathbf{u} - \mathbf{B}\mathbf{p} = \mathbf{F}\,, \tag{8}$$

$$\mathbf{P} = \widetilde{\mathbf{B}}\mathbf{u} - \mathbf{D}\mathbf{p} + \mathbf{P}^{*} \tag{9}$$

$$\varphi(\mathbf{P}) \equiv \widetilde{\mathbf{N}}\mathbf{P} - \mathbf{R} \le \mathbf{0},\ \dot{\lambda} \ge \mathbf{0},\ \widetilde{\varphi}\dot{\lambda} = 0,\ \widetilde{\dot{\varphi}}\dot{\lambda} = 0 \tag{10}$$

$$\mathbf{p} = \int_0^t \mathbf{N}\dot{\lambda}(\tau)\,d\tau \tag{11}$$

where $\mathbf{K} = \widetilde{\mathbf{C}}_0\mathbf{D}^e\mathbf{C}_0$ is the external stiffness matrix, $\mathbf{C}_0 = (\mathbf{C}_2 - \mathbf{H}\mathbf{C}_1)$, $\mathbf{B} = \widetilde{\mathbf{C}}_0\mathbf{D}^e\mathbf{H}^p$ the well-known pseudo-force matrix and $\mathbf{F} = \hat{\mathbf{F}} + \widetilde{\mathbf{C}}_1\widetilde{\mathbf{H}}^*\mathbf{f}^* + \widetilde{\mathbf{C}}_0\mathbf{D}^e\mathbf{q}_2^*$ the equivalent nodal force vector. Furthermore, $\mathbf{D} = \widetilde{\mathbf{H}}^p\mathbf{D}^e\mathbf{H}^p$ is the block diagonal stiffness matrix related to the plastic hinge sections and $\mathbf{P}^* = \widetilde{\mathbf{H}}^{p*}\mathbf{f}^* - \widetilde{\mathbf{H}}^p\mathbf{D}^e\mathbf{q}_2^*$ the generalized stress vector related to the same sections but due to mechanical and kinematical actions on the beam element.

Supposing now that the load is defined as a combination of fixed mechanical load $\mathbf{F}_0$ and cyclic mechanical and/or kinematical load $\mathbf{F}_c$ and that the cyclic load identifies with a convex polygonal shaped loading path with vertices corresponding to a set of an even number b of mutually independent load vectors, $\mathbf{F}_{ci}$, $i \in I(b) \equiv \{1,2,...,b\}$ and assuming that the cyclic load is a perfect one, i.e. for each basic load condition an opposite one exists in the load space, the steady-state response of the structure possesses the same periodicity features as the cyclic loads and it is independent of the initial conditions and of the chosen loading path. Moreover, for each cycle of the loading history, the steady-state response just depends on the sequence of the b amplified basic load conditions $\mathbf{F}_i = \xi_0\mathbf{F}_0 + \xi_c\mathbf{F}_{ci}$, being $\xi_0 > 0$ and $\xi_c > 0$ the load multipliers. As a consequence, the steady-state structural response in the cycle can be obtained by an analysis effected just for the b basic load conditions. If vector $\mathbf{Y}_i$ represents the plastic activation intensities related to the i*th* basic load condition, such as $\mathbf{p}_i = \mathbf{N}\mathbf{Y}_i$ is the plastic strain due to the i*th* basic load, and affecting the loads $\mathbf{F}_i$ by a further load multiplier $\xi > 0$, the equations governing the elastic plastic steady-state response of the structure read:

$$\mathbf{P}_i = \widetilde{\mathbf{B}}\mathbf{u}_i - \mathbf{D}\mathbf{p}_i + \xi\,\mathbf{P}_i^*, \quad \forall\, i \in I(b) \tag{12}$$

$$\mathbf{K}\mathbf{u}_i - \mathbf{B}\mathbf{p}_i = \xi\,\mathbf{F}_i, \quad \forall\, i \in I(b) \tag{13}$$

$$\mathbf{Z}_i \equiv -\varphi_i = \mathbf{R} - \xi\widetilde{\mathbf{N}}\left(\widetilde{\mathbf{B}}\mathbf{K}^{-1}\mathbf{F}_i + \mathbf{P}_i^*\right) + \mathbf{S}\mathbf{Y}_i \ge \mathbf{0}, \quad \mathbf{Y}_i \ge \mathbf{0}, \quad \widetilde{\mathbf{Z}}_i\mathbf{Y}_i = 0, \quad \forall\, i \in I(b) \tag{14}$$

where $\mathbf{S} = -\widetilde{\mathbf{N}}\left(\widetilde{\mathbf{B}}\mathbf{K}^{-1}\mathbf{B} - \mathbf{D}\right)\mathbf{N}$ is a positive semidefinite symmetric matrix. If eqs. (12-14) admit an unbounded solution $\mathbf{Y}_i$, then instantaneous collapse occurs; if they admit a finite no vanishing solution $\mathbf{Y}_i$, then the structure exhibits an elastic plastic behavior, finally, if they admit a vanishing solution $\mathbf{Y}_i$, then the full structure is elastic. Even in this last case an unknown amount of plastic strain related to the elastic shakedown can be present in the structure. An assessment over the amount of such plastic strain can be provided by means of the so-called bounding theorems (see, e.g., [8]). For the purposes of the present paper a suitable measure of the plastic deformation, related to the elastic shakedown, will be bounded as indicated hereafter:

$$\left|b^{S}\right| \leq \frac{1}{2\omega}\tilde{\mathbf{Q}}_2^{S}\mathbf{\Phi}^{e}\mathbf{Q}_2^{S} \tag{15}$$

where $b^{S}$ is the quantity to be bounded, $\omega > 0$ the perturbation multiplier and $\mathbf{Q}_2^{S}$ the self stress field related to the perturbed yield domain.

## Minimum volume design formulation

Making reference to the structure previously described, the minimum volume design formulation, according simultaneously to an elastic shakedown criterion and to a plastic shakedown one, and taking into account suitable limits on the plastic deformations at the limit state of the plastic shakedown as well as on the plastic strain related to the elastic shakedown, reads as follows:

$$\min_{\left(\mathbf{d},\mathbf{Q}_2^{F},\omega,\ \mathbf{Q}_2^{S}\right)} V \tag{16a}$$

$$\mathbf{d} - \bar{\mathbf{d}} \geq \mathbf{0} \tag{16b}$$

$$\mathbf{Z}_i = \mathbf{R} - \xi\xi_c^{F}\tilde{\mathbf{N}}\left(\tilde{\mathbf{B}}\mathbf{K}^{-1}\mathbf{F}_{ci} + \mathbf{P}_{ci}^{*}\right) + \mathbf{S}\mathbf{Y}_i \geq \mathbf{0}\,, \quad \mathbf{Y}_i \geq \mathbf{0}\,, \quad \tilde{\mathbf{Y}}_i\,\mathbf{Z}_i = 0 \tag{16c}$$

$$\sum_{i=1}^{b}\mathbf{M}_i\mathbf{Y}_i - \mathbf{b}^{F} \leq \mathbf{0}\,, \qquad -\sum_{i=1}^{b}\mathbf{M}_i\mathbf{Y}_i - \mathbf{b}^{F} \leq \mathbf{0} \tag{16d}$$

$$-\varphi_i^{F} \equiv \mathbf{Z}_i - \xi\xi_0^{F}\tilde{\mathbf{N}}\left(\tilde{\mathbf{B}}\mathbf{K}^{-1}\mathbf{F}_0 + \mathbf{P}_0^{*}\right) - \tilde{\mathbf{N}}\tilde{\mathbf{H}}^{p}\mathbf{Q}_2^{F} \geq \mathbf{0} \tag{16e}$$

$$\tilde{\mathbf{C}}_0\mathbf{Q}_2^{F} = \mathbf{0} \tag{16f}$$

$$-\hat{\varphi}_i^{S} \equiv \mathbf{R} - \omega\hat{\mathbf{R}} - \xi\xi_0^{S}\tilde{\mathbf{N}}\left(\tilde{\mathbf{B}}\mathbf{K}^{-1}\mathbf{F}_0 + \mathbf{P}_0^{*}\right) - \xi\xi_c^{S}\tilde{\mathbf{N}}\left(\tilde{\mathbf{B}}\mathbf{K}^{-1}\mathbf{F}_{ci} + \mathbf{P}_{ci}^{*}\right) - \tilde{\mathbf{N}}\tilde{\mathbf{H}}^{p}\mathbf{Q}_2^{S} \geq \mathbf{0} \tag{16g}$$

$$\tilde{\mathbf{C}}_0\mathbf{Q}_2^{S} = \mathbf{0} \tag{16h}$$

$$\frac{1}{2\omega}\tilde{\mathbf{Q}}_2^{S}\mathbf{\Phi}^{e}\mathbf{Q}_2^{S} - \bar{b}^{S} \leq 0 \tag{16i}$$

$$\omega > 0 \tag{16j}$$

where $\bar{\mathbf{d}}$ are the minimal values of the design variables $\mathbf{d}$, $\mathbf{Q}_2^{F}$ the self stress field at the limit state of the plastic shakedown, $\hat{\mathbf{R}}$ the perturbation vector, $\hat{\varphi}_i^{S}$ the perturbed yield domain. Eqs. (16c) represent the elastic plastic steady-state response of the structure subjected just to the action of the purely cyclic load amplified by $\xi\xi_c^{F}$; by eqs. (16d) given limits are imposed to suitable measures (depending on matrices $\mathbf{M}_i$) of the plastic deformation at the limit state of plastic shakedown; eqs. (16e,f) represent the limit conditions for plastic shakedown (for fixed loads amplified by $\xi\xi_0^F$), while eqs. (16g,h) represent the limit conditions for elastic shakedown related to a perturbed yield function, being $\xi\xi_0^S$ and $\xi\xi_c^S$ the fixed and cyclic load multipliers, respectively. With eqs. (16i,j), a given limit to the bound on a chosen measure of the plastic deformation related to the elastic shakedown is imposed. Eqs. (16c,e,g) must be satisfied $\forall\, i \in I(b)$.

The solution to problem (16) can be pursued by operating parametrically, $\omega$ being the parameter [9], and utilizing suitable iterative techniques [3]. Therefore, at first the following problem must be solved, denoting with $\overline{\omega}$ the fixed value parameter:

$$\min_{\left(\mathbf{d},\mathbf{Q}_2^F,\,\mathbf{Q}_2^S\right)} V \tag{17a}$$

$$\mathbf{d} - \overline{\mathbf{d}} \geq \mathbf{0} \tag{17b}$$

$$\mathbf{Z}_i = \mathbf{R} - \xi\xi_c^F \widetilde{\mathbf{N}}\left(\widetilde{\mathbf{B}}\mathbf{K}^{-1}\mathbf{F}_{ci} + \mathbf{P}_{ci}^*\right) + \mathbf{S}\,\mathbf{Y}_i \geq \mathbf{0}\,,\quad \mathbf{Y}_i \geq \mathbf{0}\,,\quad \widetilde{\mathbf{Y}}_i\,\mathbf{Z}_i = 0 \tag{17c}$$

$$\sum_{i=1}^{b} \mathbf{M}_i\mathbf{Y}_i - \mathbf{b}^F \leq \mathbf{0}\,, \qquad -\sum_{i=1}^{b} \mathbf{M}_i\mathbf{Y}_i - \mathbf{b}^F \leq \mathbf{0} \tag{17d}$$

$$-\varphi_i^F \equiv \mathbf{Z}_i - \xi\xi_0^F \widetilde{\mathbf{N}}\left(\widetilde{\mathbf{B}}\mathbf{K}^{-1}\mathbf{F}_0 + \mathbf{P}_0^*\right) - \widetilde{\mathbf{N}}\widetilde{\mathbf{H}}^p\mathbf{Q}_2^F \geq \mathbf{0} \tag{17e}$$

$$\widetilde{\mathbf{C}}_0\mathbf{Q}_2^F = \mathbf{0} \tag{17f}$$

$$-\hat{\varphi}_i^S \equiv \mathbf{R} - \overline{\omega}\hat{\mathbf{R}} - \xi\xi_0^S \widetilde{\mathbf{N}}\left(\widetilde{\mathbf{B}}\mathbf{K}^{-1}\mathbf{F}_0 + \mathbf{P}_0^*\right) - \xi\xi_c^S \widetilde{\mathbf{N}}\left(\widetilde{\mathbf{B}}\mathbf{K}^{-1}\mathbf{F}_{ci} + \mathbf{P}_{ci}^*\right) - \widetilde{\mathbf{N}}\widetilde{\mathbf{H}}^p\mathbf{Q}_2^S \geq \mathbf{0} \tag{17g}$$

$$\widetilde{\mathbf{C}}_0\mathbf{Q}_2^S = \mathbf{0} \tag{17h}$$

holding eqs. (17c,e,g) $\forall\, i \in I(b)$. The optimal value of $\omega$ can be obtained by solving:

$$\min_{(\omega)} V(\omega) \tag{18a}$$

$$\frac{1}{2\omega}\widetilde{\mathbf{Q}}_2^S\mathbf{\Phi}^e\mathbf{Q}_2^S - \overline{b}^S \leq 0 \tag{18b}$$

$$\omega > 0 \tag{18c}$$

Solving the above sequence of minimum problems (17, 18) a very good design can be determined depending on the accuracy in performing the sensitivity analysis of the quantities appearing in problem (17) as function of $\omega$.

## Conclusions

The minimum volume design of elastic perfectly plastic frames subjected to a combination of fixed and cyclic loads has been studied. The search problem has been formulated so that the optimal structure is able to elastically adapt to the loads in serviceability conditions and to prevent the incremental/instantaneous collapse for very high intensity loads. Furthermore, suitable limits on the plastic deformations occurring at the limit state of the plastic shakedown have been imposed as well as a chosen measure of the plastic deformation related to the elastic shakedown has been suitably bounded.

The proposed minimum problem is a strongly non linear mathematical programming one and, as a consequence, special emphasis has been devoted in the present paper to the proposing of a suitable solution procedure. In the computational stage several applications devoted to steel frames have been effected, choosing different kinematical quantities to introduce within the constraints of the problem in order to control the plastic deformations related to the structure behavior at the limit states of elastic and plastic shakedown. In all the examined cases, the obtained results, have been encouraging and characterized by good coherence, requiring in addition a not very high computational effort.

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