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Further developments on GMLS: plates on elastic foundation

Carlos Tiago[1], Vitor M.A. Leitão[1]

Summary

The Moving Least Squares (MLS) approximation has been used with remarkable success as basis of some meshless methods. In this work the GMLS (Generalized Moving Least Squares) is used in the EFG context for the numerical solution of thin plates on elastic foundation. The main advantage of the GMLS approximation is the improvement on the quality and accuracy of the approximation when compared to MLS approximations using similar number of nodes.

Introduction

The MLS (Moving Least Squares) approximation has been used (since the "diffuse elements" were first presented in 1992 by Nayroles *et al* [1]) as the basis of several meshless methods. These include (but are not restricted to) Element-Free Galerkin method [2], the *hp*-cloud method [3] (which uses enrichment over the partition of unity generated by the zero-th order MLS approximation, the Sheppard function), the node-by-node meshless method [4] and the Meshless Local Petrov-Galerkin (MLPG) method [5] (although the use of MLS is not mandatory, as it was show by Atluri *et al* [6]). The GMLS was presented, in 1999 by Atluri *et al* [7]) in the context of meshless local methods and it was applied to one-dimensional bending problems of Euler-Bernoulli beams. Further developments on the method were presented by Raju *et al* [8]), where a computationally less expensive approach that eliminate the domain integrals for the stiffness part (in general, the evaluation of the force vector still requires integrations on the domain) was presented. Application of the GMLS approximation to two dimensional problems was first proposed (to the authors knowledge) by the authors [9] in 2003. The performance of the method using quadratic, cubic and quartic polynomial basis was analyzed in plate bending problems using the EFG method. Further work on the subject is also being presented elsewhere [10]. Comparisons with the MLS are presented herein which have revealed an increase of the accuracy of the GMLS for the same number of degrees of freedom. It is very clear that, for the same number of nodes, the accuracy is higher and the converge is faster. This is by no means a surprise, because three degrees of freedom per node are used by the GMLS, whereas only one is required by the MLS.

The use of MLS approximation allows for a quite convenient generation of continuous functions of arbitrary order using only nodal values. Several works have been presented showing the success of the approach for solving static and free vibration of thin plate bending problems by different methods: the Element-Free Galerkin [11], [12], the *Hp* Clouds [13] and MLPG [14], [15].

Following the Finite Element Method (FEM) approach, it seems rather natural to utilize, for bending problems, other degrees of freedom besides the generalized displacement

[1]DECivil-ICIST, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

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at the nodes. The basic idea of the GMLS is to build from a purely unstructured mesh of points an approximation using not only the value of the function but also of its derivatives at the nodes.

The variational form of the equilibrium equations will be presented for thin plates on elastic foundation followed by the presentation of the governing system of equations. Implementation aspects are discussed followed by the numerical results obtained and the consequent conclusions.

Variational form of the equilibrium equations

The governing differential equation of the problem is:

$$\nabla^4 w + K_w w = \overline{p} \tag{1}$$

where w is the deflection of the middle surface of the plate, K_w is the Winkler modulus and $\overline{p}$ is the imposed load.

As the GMLS approximation does not satisfy the Kronecker delta criterion $\Phi_I(x_J) \neq \delta_{IJ}$, an extended weak form, which includes Lagrange multipliers functions, of the problem is used.

Consider the following fields:

1. the approximation and the weighting functions $w(\mathbf{x}) \in H^2(\Omega)$ and $\delta w(\mathbf{x}) \in H^2(\Omega)$ for the displacements, w, on the domain, Ω;
2. the approximation and weighting functions, $\lambda_{M_n}(\mathbf{x}) \in H^0(\Gamma_{\overline{\frac{\partial w}{\partial n}}})$ and $\delta\lambda_{M_n}(\mathbf{x}) \in H^0(\Gamma_{\overline{\frac{\partial w}{\partial n}}})$, of the normal bending moment, M_n, in the essential boundary where the normal rotation, $\frac{\partial w}{\partial n}$, is prescribed;
3. the approximation and weighting functions, $\lambda_{V_n}(\mathbf{x}) \in H^0\Gamma_{\overline{w}}$ and $\delta\lambda_{V_n}(\mathbf{x}) \in H^0(\Gamma_{\overline{w}})$, of the effective normal shear force, V_n, in the essential boundary where the transverse displacement is prescribed, $\Gamma_{\overline{w}}$.

Here H^m denote the Sobolev space of degree m. Then, the solution of the problem is the same as the solution expressed by the following equation

$$\int_\Omega \delta\chi^{\mathrm{T}}\mathbf{D}\chi \mathrm{d}\Omega + \int_\Omega \delta w K_w w \mathrm{d}\Omega - \int_\Omega \delta w \overline{p}\, \mathrm{d}\Omega - \int_{\Gamma_{\overline{M}_n}} \overline{M}_n \left(-\frac{\partial \delta w}{\partial n}\right) \mathrm{d}\Gamma_{\overline{M}_n} -$$
$$\int_{\Gamma_{\overline{V}_n}} \overline{V}_n \delta w \mathrm{d}\Gamma_{\overline{V}_n} - \int_{\Gamma_{\overline{\frac{\partial w}{\partial n}}}} \delta\lambda_{M_n} \left(-\frac{\partial w}{\partial n} + \overline{\frac{\partial w}{\partial n}}\right) \mathrm{d}\Gamma_{\overline{\frac{\partial w}{\partial n}}} - \int_{\Gamma_{\overline{w}}} \delta\lambda_{V_n} (w - \overline{w})\, \mathrm{d}\Gamma_{\overline{w}} -$$
$$\int_{\Gamma_{\overline{\frac{\partial w}{\partial n}}}} \lambda_{M_n} \left(-\frac{\partial \delta w}{\partial n}\right) \mathrm{d}\Gamma_{\overline{\frac{\partial w}{\partial n}}} - \int_{\Gamma_{\overline{w}}} \lambda_{V_n} \delta w \mathrm{d}\Gamma_{\overline{w}} = 0 \tag{2}$$

where $\chi = \mathbf{L}^{\Omega} w$, $M_n = \mathrm{L}^{M_n} w$ and $V_n = \mathrm{L}^{V_n} w$. In these expressions the curvatures, χ, the normal bending moment and the normal effective shear force are derived from the displacement field through the application of the differential operators $\mathbf{L}^{\Omega}$, L^{M_n} and L^{V_n}, respectively. Prescribed quantities are denoted by "$\overline{\quad}$".

Governing System

The GMLS approximation will not be revisited here. The functional which is to be locally minimized and the resulting expressions were presented before by Atluri *et al* [7]). Examples of the resulting nodal functions are presented by the authors [10] elsewhere.

The GMLS can assume the convenient form

$$w(\mathbf{x}) = \Phi^{w}(\mathbf{x})\mathbf{U} \qquad \delta w(\mathbf{x}) = \Phi^{w}(\mathbf{x})\delta\mathbf{U} \tag{3}$$

The Lagrange functions on the essential boundary are discretized as follows

$$\lambda_{M_n}(\mathbf{x}) = \Phi^{\lambda}(\mathbf{x})\Lambda_{M_n} \qquad \lambda_{V_n}(\mathbf{x}) = \Phi^{\lambda}(\mathbf{x})\Lambda_{V_n} \tag{4a}$$

$$\delta\lambda_{M_n}(\mathbf{x}) = \Phi^{\lambda}(\mathbf{x})\delta\Lambda_{M_n} \qquad \delta\lambda_{V_n}(\mathbf{x}) = \Phi^{\lambda}(\mathbf{x})\delta\Lambda_{V_n} \tag{4b}$$

In expressions (3) and (4) the quantities $\mathbf{U}$, $\delta\mathbf{U}$, Λ_{M_n}, $\delta\Lambda_{M_n}$, Λ_{V_n} and $\delta\Lambda_{V_n}$ are the vectors that collect the discrete parameters related to the respective continuous fields they represent.

Using the approximations (3) for w and δw and the expressions (4) on (2), for arbitrary variations of $\delta\mathbf{U}$, $\delta\Lambda_{M_n}$ and $\delta\Lambda_{V_n}$, the discretized problem can assume the form

$$\begin{bmatrix} \mathbf{K}+\mathbf{K}_w & \mathbf{G}^{w} & \mathbf{G}^{\frac{\partial w}{\partial n}} \\ \mathbf{G}^{wT} & & \\ \mathbf{G}^{\frac{\partial w}{\partial n}T} & & \end{bmatrix} \begin{Bmatrix} \mathbf{U} \\ \Lambda_{M_n} \\ \Lambda_{V_n} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ \mathbf{q}^{w} \\ \mathbf{q}^{\frac{\partial w}{\partial n}} \end{Bmatrix}, \tag{5}$$

where the meaning of all of the quantities is detailed in [13] except $\mathbf{K}_w$ which is given by

$$\mathbf{K}_w = \int_{\Omega} \Phi^{T} K_w \Phi \mathrm{d}\Omega. \tag{6}$$

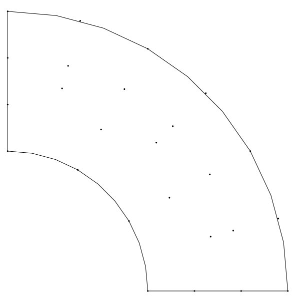
Implementation issues

As meshless methods are still very recent, several aspects in its implementation still need to be investigated. The weight function used in the present work is given in [7] because it allows control on the continuity of the function. Here $s = 4$ was used ensuring continuous third derivatives and, consequently, continuous shear forces. The radius of support was kept constant for all the nodes. For the basis $\mathbf{p}$ complete quadratic, cubic and quartic polynomials were used. The integrations on the domain were carried out using a background integration cell structure. This was not necessary at all, but it simplifies the way the integration of the weak form was done. Gauss-Legendre quadrature was carried out using 6×6 points.

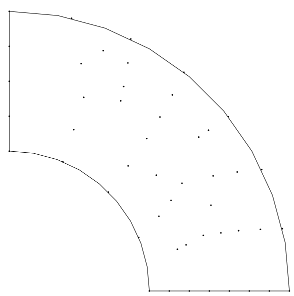
The MLS approximation was used for the functions $\Phi^{\lambda}(\mathbf{x})$. This seams to be more consistent with the approximations made on the domain than with the usual linear Lagrange interpolation functions.

Numerical example

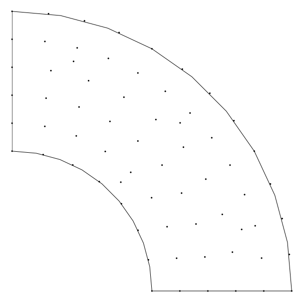
Consider a simply supported circular plate on a elastic foundation with a central hole and subjected to a uniform load. The data is: Young's modulus $E = 30.0 \cdot 10^6$, Poisson's ratio $\nu = 0.3$, thickness $t = 0.1$, Winkler's foundation modulus $K_w = 1.0 \cdot 10^6$, exterior radius $r_e = 1.0$, interior radius $r_e = 0.5$ and uniform load $p = 1.0$. Double symmetry was used to build up the model. The exact geometry of the problem was considered by the inclusion of two circular sides. The three meshes represented on figure 1 were used. The numerical exact solution was generated by the solution of the analogous one-dimensional problem written in polar coordinates and using the Range-Kutta method. The accuracy of



(a) Coarse (25 nodes).



(b) Medium (48 nodes).



(c) Fine (67 nodes).

Figure 1: Meshes used in the analysis.

the solutions was measured using the relative error of strain energy, ε_U, given by

$$\varepsilon_U = \left| \frac{U_{\text{num}} - U_{\text{exact}}}{U_{\text{exact}}} \right| \qquad \text{where} \qquad U_\square = \frac{1}{2} \int_\Omega \chi_\square^T \mathbf{D} \chi_\square \, d\Omega. \tag{7}$$

The results obtained are represented in figure 2 for the three basis and the three meshes. The solution obtained for the radial moment, m_{rr}, and the radial shear force, q_r, using the coarse mesh and quadratic basis is represented in figure 3.

Conclusions

A EFG procedure for thin plates on elastic foundation which uses GMLS as approximation was presented in this work. The Lagrange multiplier method was used to imposed the essential boundary conditions and the approximation of the corresponding reaction forces was made by one-dimensional MLS. The performance of the procedure was compared with the traditional MLS. As expected, the GMLS provides a superior accuracy over MLS for the same number of nodes. This fact becomes very clear by observing figure 3. The explanation is because GMLS resorts to three times more degrees of freedom per node than MLS. But even for the same number of the degrees of freedom, the GMLS provides a much better solution in the example presented. In the implementation done it was also

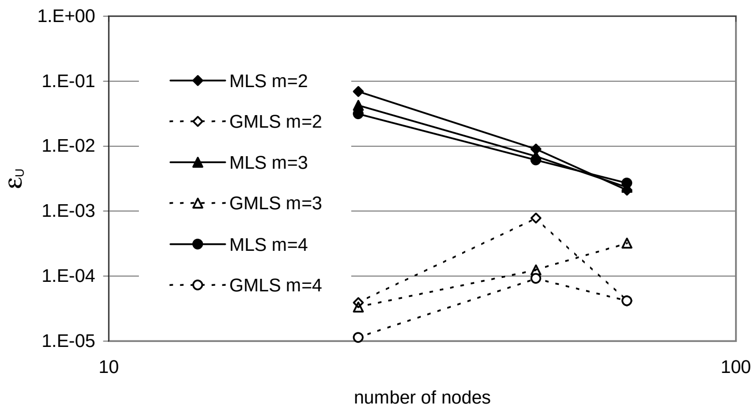


Figure 2: Results for the circular plate.

observed that, for the same basis, integration rule and number of degrees of freedom, the GMLS requires less CPU time.

Acknowledgments

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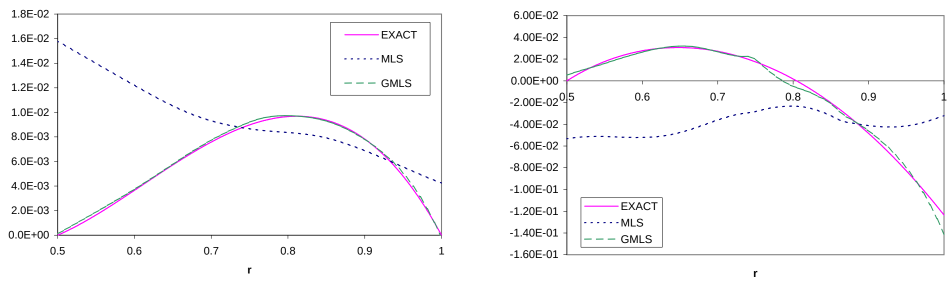

(a) Radial moment, m_{rr}. (b) Radial shear force, q_r.

Figure 3: Solution obtained for the coarse mesh (25 nodes) using quadratic basis.

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