Strain-gradient elasticity: numerical methods, a reciprocal theorem and a Saint-Venant type principle

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Summary

Theories with intrinsic or material length scales find applications in the modeling of size-dependent phenomena, such as the localization of plastic flow into shear bands. In gradient-type plasticity theories, length scales are introduced through the coefficients of spatial gradients of one or more internal variables. In elasticity, length scales enter the constitutive equations through the elastic strain energy function, which, in this case, depends not only on the strain tensor but also on gradients of the rotation and strain tensors. In the present paper we focus our attention on the strain-gradient elasticity theories developed by Mindlin and co-workers in the 1960's. In such theories, when the problem is formulated in term of displacements, the governing partial differential equation is of fourth order. If traditional finite elements are used for the numerical solution of such problems, then C^1 displacement continuity is required. An alternative "mixed" finite element formulation is developed, in which the displacement and displacement –gradients are used as independent unknowns and their relationship is enforced in an "integral-sense". The resulting finite elements require only C^0 continuity and are simple to formulate.

Introduction

Classical (local) continuum constitutive models possess no material/intrinsic length scale. The typical dimensions of length that appear are associated with the overall geometry of the domain under consideration. In spite of the fact that classical theories are quite sufficient for most applications, there is ample experimental evidence, which indicates that, in certain applications, there is significant dependence on additional length/scale parameters. A first attempt to incorporate length scale effects in elasticity was made by Mindlin [4] and Koiter [3]. More recently, a variety of 'gradient-type' theories have been used in order to introduce material length scales into constitutive models. In the following we summarize briefly a family of strain-gradient elasticity theories introduced by Mindlin and co-workers [5, 6] and present a variational formulation, which is used together with the finite element method for the numerical solution of boundary value problems.

A review of strain-gradient elasticity theories

Let $\mathbf{u}$ be the displacement field. The following quantities are defined:

$$\varepsilon_{ij} = u_{(i,j)} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \text{strain}\,, \quad \Omega_{ij} = u_{[i,j]} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) = -e_{ijk}\omega_k = \text{rotation tensor}\,,$$

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$$\omega_i = \frac{1}{2}(\nabla \times \mathbf{u})_i = \frac{1}{2} e_{ijk} u_{k,j} = -\frac{1}{2} e_{ijk} \Omega_{jk} = \text{rotation vector (axial vector of } \mathbf{\Omega}),$$

$$\bar{\kappa}_{ij} = \omega_{j,i} = \text{rotation gradient}, \quad \bar{\kappa}_{ii} = 0,$$

$$\tilde{\kappa}_{ijk} = u_{k,ij} = \varepsilon_{jk,i} + \varepsilon_{ki,j} - \varepsilon_{ij,k} = \tilde{\kappa}_{jik} = \text{second gradient of displacement},$$

$$\hat{\kappa}_{ijk} = \frac{1}{2}\left(u_{j,ki} + u_{k,ji}\right) = \varepsilon_{jk,i} = \hat{\kappa}_{ikj} = \text{strain gradient},$$

$$\bar{\bar{\kappa}}_{ijk} = \frac{1}{3}\left(u_{i,jk} + u_{j,ki} + u_{k,ij}\right) = \frac{1}{3}\left(\varepsilon_{ij,k} + \varepsilon_{jk,i} + \varepsilon_{ki,j}\right) = \bar{\bar{\kappa}}_{jik} = \bar{\bar{\kappa}}_{ikj} = \bar{\bar{\kappa}}_{kji} = \text{symmetric part of } \tilde{\kappa}_{ijk} \text{ or } \hat{\kappa}_{ijk}$$

where e_{ijk} is the alternating symbol. The above quantities are related by the following expressions (Mindlin and Eshel [5]):

$$\tilde{\kappa}_{ijk} = \hat{\kappa}_{ijk} + \hat{\kappa}_{jki} - \hat{\kappa}_{kij} = \bar{\bar{\kappa}}_{ijk} + \frac{2}{3}\bar{\kappa}_{ip} e_{pjk} + \frac{2}{3}\bar{\kappa}_{jp} e_{pik},$$

$$\hat{\kappa}_{ijk} = \frac{1}{2}\left(\tilde{\kappa}_{ijk} + \tilde{\kappa}_{ikj}\right) = \bar{\bar{\kappa}}_{ijk} - \frac{1}{3}\bar{\kappa}_{jp} e_{kip} - \frac{1}{3}\bar{\kappa}_{kp} e_{jip},$$

$$\bar{\kappa}_{ij} = \frac{1}{2}\tilde{\kappa}_{ipk}\varepsilon_{jpk} = \hat{\kappa}_{pik}\varepsilon_{jpk}, \qquad \bar{\bar{\kappa}}_{ijk} = \frac{1}{3}\left(\tilde{\kappa}_{ijk} + \tilde{\kappa}_{jki} + \tilde{\kappa}_{kij}\right) = \frac{1}{3}\left(\hat{\kappa}_{ijk} + \hat{\kappa}_{jki} + \hat{\kappa}_{kij}\right).$$

The alternative forms of the strain-gradient elasticity theory given by Mindlin [4] are summarized in the following. The strain energy density W is written in three equivalent forms: $W = \tilde{W}(\varepsilon, \tilde{\kappa}) = \hat{W}(\varepsilon, \hat{\kappa}) = \bar{W}(\varepsilon, \bar{\kappa}, \bar{\bar{\kappa}})$. Mindlin refers to the description $W = \tilde{W}(\varepsilon, \tilde{\kappa})$ as "Type I", to $W = \hat{W}(\varepsilon, \hat{\kappa})$ as "Type II", and to $W = \bar{W}(\varepsilon, \bar{\kappa}, \bar{\bar{\kappa}})$ as "Type III". Using the above forms of the elastic strain energy density, one defines the following quantities:

$$\bar{\sigma}_{ij} = \frac{\partial \tilde{W}}{\partial \varepsilon_{ij}} = \frac{\partial \hat{W}}{\partial \varepsilon_{ij}} = \frac{\partial \bar{W}}{\partial \varepsilon_{ij}} = \bar{\sigma}_{ji}, \qquad \tilde{\mu}_{ijk} = \frac{\partial \tilde{W}}{\partial \tilde{\kappa}_{ijk}} = \tilde{\mu}_{jik}, \qquad \hat{\mu}_{ijk} = \frac{\partial \hat{W}}{\partial \hat{\kappa}_{ijk}} = \hat{\mu}_{ikj},$$

$$\bar{\mu}_{ij} = \frac{\partial \bar{W}}{\partial \bar{\kappa}_{ij}}, \qquad \bar{\bar{\mu}}_{ijk} = \frac{\partial \bar{W}}{\partial \bar{\bar{\kappa}}_{ijk}} = \bar{\bar{\mu}}_{jik} = \bar{\bar{\mu}}_{ikj} = \bar{\bar{\mu}}_{kji}.$$

Variational formulation

A given boundary value problem in strain-gradient elasticity can be formulated in any of the three equivalent ways discussed in the previous section. Here we discuss the Type III formulation and emphasize the calculation of true stresses and true couple stresses. The governing equations in the volume V of the body are:

$$\sigma_{ji,j} = 0, \quad \sigma_{ij} = \bar{\sigma}_{ij} + \bar{\sigma}^{(2)}_{ij}, \quad \bar{\sigma}^{(2)}_{ij} = -\bar{\bar{\mu}}_{ijk,k} - \frac{1}{2} e_{ijk}\bar{\mu}_{pk,p},$$

$$\varepsilon_{ij} = u_{(i,j)}, \quad \omega_i = -\frac{1}{2} e_{ijk} u_{j,k}, \quad \bar{\kappa}_{ij} = \omega_{j,i}, \quad \bar{\bar{\kappa}}_{ijk} = \frac{1}{3}\left(\varepsilon_{ij,k} + \varepsilon_{jk,i} + \varepsilon_{ki,j}\right),$$

$$\bar{\sigma}_{ij} = \frac{\partial \bar{W}}{\partial \varepsilon_{ij}}, \quad \bar{\mu}_{ij} = \frac{\partial \bar{W}}{\partial \kappa_{ij}}, \quad \bar{\bar{\mu}}_{ijk} = \frac{\partial \bar{W}}{\partial \bar{\bar{\kappa}}_{ijk}}.$$

The corresponding boundary conditions are:

$$u_i = \bar{u}_i \text{ on } S_u, \; P_i \equiv n_j \sigma_{ji} - \frac{n_j \bar{\mu}^n_{,k} e_{ijk}}{2} + \left[\left(D_p n_p \right) n_j - D_j \right] \left(n_k \bar{\bar{\mu}}_{kji} + n_i n_p n_k \bar{\bar{\mu}}_{kpj} \right) = \bar{P}_i \text{ on } S_P, \quad (1)$$

$$\omega^t_i = \bar{\omega}^t_i \text{ on } S_\omega, \qquad Q^t_i \equiv n_j \bar{\mu}^t_{ji} + 2 n_q n_j n_k \bar{\bar{\mu}}_{kjp} e_{qpi} = \bar{Q}^t_i \quad \text{on } S_Q, \quad (2)$$

$$n_i n_j \varepsilon_{ij} = \bar{\varepsilon} \quad \text{on } S_\varepsilon, \qquad R \equiv n_i n_j n_k \bar{\bar{\mu}}_{ijk} = \bar{R} \text{ on } S_R, \quad (3)$$

$$u_i = \bar{u}^\alpha_i \quad \text{on } C^\alpha_u, \qquad E^\alpha_i \equiv [[\frac{1}{2} s_i \bar{\mu}^n + \ell_j n_k \left(\bar{\bar{\mu}}_{kji} + n_i n_p \bar{\bar{\mu}}_{pjk} \right)]] = \bar{E}^\alpha_i \text{ on } C^\alpha_E, \quad (4)$$

where $\omega^t_i = \omega_i - \omega_j n_j n_i$, $\bar{\mu}^n = n_i n_j \bar{\mu}_{ij}$, $\bar{\mu}^t_{ij} = \bar{\mu}_{ij} - \bar{\mu}_{ik} n_k n_j$, $DA = n_i A_{,i}$, $D_i A = A_{,i} - n_i DA$, $S_u \cup S_P = S$, $S_\omega \cup S_Q = S$, $S_\varepsilon \cup S_R = S$, $C^\alpha_u \cup C^\alpha_E = C^\alpha$, $S_u \cap S_P = \varnothing$, $S_\omega \cap S_Q = \varnothing$, $S_\varepsilon \cap S_R = \varnothing$, $C^\alpha_u \cap C^\alpha_E = \varnothing$, and $\left(\bar{\mathbf{u}}, \bar{\boldsymbol{\omega}}^t, \bar{\varepsilon}, \bar{\mathbf{u}}^\alpha \right)$ are known functions. In the above expressions, S is the surface of the body under consideration; when the outer surface S is piecewise smooth, it can be divided into a finite number of smooth surfaces S^α $\left(\alpha = 1, 2, \ldots \right)$ each bounded by an edge C^α. In the above expressions, the double brackets [[ ]] indicate the jump in the value of the enclosed quantity across C^α, and $\mathbf{l} = \mathbf{s} \times \mathbf{n}$, where $\mathbf{s}$ is the unit vector tangent to C^α. The generalized loads $\left(\mathbf{P}, \mathbf{Q}^t, R, \mathbf{E}^\alpha \right)$ are defined in equations (1)-(4) and take the prescribed values $\left(\bar{\mathbf{P}}, \bar{\mathbf{Q}}^t, \bar{R}, \bar{\mathbf{E}}^\alpha \right)$ on the corresponding parts of the surface S.

Amanatidou and Aravas [1] have shown that the solution of the problem can be given by the stationarity condition $\delta \Pi = 0$ of the functional

$$\Pi\left(\mathbf{u}, \boldsymbol{\omega}, \boldsymbol{\varepsilon}, \bar{\boldsymbol{\sigma}}^{(2)} \right) = \int_V W\left(u_{(i,j)}, \bar{\boldsymbol{\kappa}}(\boldsymbol{\omega}), \bar{\bar{\boldsymbol{\kappa}}}(\boldsymbol{\varepsilon}) \right) dV + \int_V \left[u_{i,j} - \left(\varepsilon_{ij} - e_{ijk} \omega_k \right) \right] \bar{\sigma}^{(2)}_{ij} dV$$

$$- \int_{S_P} \bar{P}_i v_i dS - \int_{S_Q} \bar{Q}^t_i \omega^t_i dS - \int_{S_R} \bar{R} n_i n_j \varepsilon_{ij} dS - \sum_\alpha \oint_{C^\alpha_E} \bar{E}^\alpha_i u_i dS + \int_S \left(\frac{1}{2} e_{ijk} u_{k,j} n_i - \omega_i n_i \right) n_p \bar{\mu}_{pq} n_q dS$$

$$+ \int_S \left[u_{(i,j)} - 2 n_i n_j u_{[i,k]} - n_i n_j n_p n_q u_{(p,q)} - \left(\varepsilon_{ij} + 2 n_j n_k e_{ikp} \omega_p - n_i n_j n_p n_q \varepsilon_{pq} \right) \right] n_r \bar{\bar{\mu}}_{rij} dS,$$

where $\varepsilon_{ij} = \varepsilon_{ji}$, $\bar{\kappa}_{ij}(\boldsymbol{\omega}) = \omega_{j,i}$, $\bar{\bar{\kappa}}_{ijk}(\boldsymbol{\varepsilon}) = \frac{1}{3}\left(\varepsilon_{ij,k} + \varepsilon_{jk,i} + \varepsilon_{ki,j} \right)$, $\bar{\mu}_{ij} = \frac{\partial \bar{W}}{\partial \bar{\kappa}_{ij}}$, $\bar{\bar{\mu}}_{ijk} = \frac{\partial \bar{W}}{\partial \bar{\bar{\kappa}}_{ijk}}$, $\delta \mathbf{u} = \mathbf{0}$ on S_u and C^α_u, $\delta \boldsymbol{\omega}^t = \mathbf{0}$ on S_ω, and $n_i n_j \delta \varepsilon_{ij} = 0$ on S_ε.

Amanatidou and Aravas [1] used the above variational principle and developed a nine-node isoparametric plane strain finite element with 70 degrees of freedom (III9-70). The quantities $\left(u_1, u_2, \omega_3, \varepsilon_{11}, \varepsilon_{22}, 2\varepsilon_{12}\right)$ are used as degrees of freedom at all nodes and the quantities $\left(\sigma_{11}^{(2)}, \sigma_{22}^{(2)}, \sigma_{(12)}^{(2)}, \sigma_{[12]}^{(2)}\right)$ are additional degrees of freedom at the corner nodes. A bi-quadratic Lagrangian interpolation for $\left(u_1, u_2, \omega_3, \varepsilon_{11}, \varepsilon_{22}, 2\varepsilon_{12}\right)$ and a bi-linear for $\left(\sigma_{11}^{(2)}, \sigma_{22}^{(2)}, \sigma_{(12)}^{(2)}, \sigma_{[12]}^{(2)}\right)$ are used in the isoparametric plane.

A reciprocity theorem in linear strain-gradient elasticity and the corresponding Saint-Venant-type principle

The general form of the strain energy density function for a linear material is (Mindlin [4])

$$\hat{W}\left(\boldsymbol{\varepsilon}, \hat{\boldsymbol{\kappa}}\right) = \frac{1}{2} c_{ijpq}\, \varepsilon_{ij}\, \varepsilon_{pq} + \frac{1}{2} a_{ijkpqr}\, \hat{\kappa}_{ijk}\, \hat{\kappa}_{pqr} + g_{ijkpq}\, \hat{\kappa}_{ijk}\, \varepsilon_{pq},$$

where the constitutive tensors $\mathbf{c}$, $\mathbf{a}$ and $\mathbf{g}$ have the symmetries.

$$c_{ijpq} = c_{pqij} = c_{jipq} = c_{ijqp}, \qquad a_{ijkpqr} = a_{pqrijk} = a_{kjipqr} = a_{ijkrqp}, \qquad g_{ijkpq} = g_{ikjpq} = g_{ijkqp}.$$

The corresponding $\bar{\sigma}_{ij}$ and $\hat{\mu}_{ijk}$ are of the form

$$\bar{\sigma}_{ij} = \frac{\partial \hat{W}}{\partial \varepsilon_{ij}} = c_{ijpq}\, \varepsilon_{pq} + g_{pqrij}\, \hat{\kappa}_{pqr}, \qquad \hat{\mu}_{ijk} = \frac{\partial \hat{W}}{\partial \hat{\kappa}_{ijk}} = a_{ijkpqr}\, \hat{\kappa}_{pqr} + g_{ijkpq}\, \varepsilon_{pq}.$$

Giannakopoulos *et al.* [2] have shown that a "reciprocity" theorem exists only if $\mathbf{g} = \mathbf{0}$, in which case

$$\bar{\sigma}_{ij} = c_{ijpq}\, \varepsilon_{pq} \qquad \text{and} \qquad \hat{\mu}_{ijk} = a_{ijkpqr}\, \hat{\kappa}_{pqr}.$$

The corresponding type III form of the elastic strain energy density function is then

$$\bar{W}\left(\boldsymbol{\varepsilon}, \bar{\boldsymbol{\kappa}}, \bar{\bar{\boldsymbol{\kappa}}}\right) = \frac{1}{2} c_{ijpq} \varepsilon_{ij} \varepsilon_{pq} + \frac{1}{2} a_{ijkpqr} \left(\bar{\bar{\kappa}}_{ijk} - \frac{1}{3} \bar{\kappa}_{js} e_{kis} - \frac{1}{3} \bar{\kappa}_{ks} e_{jis} \right) \left(\bar{\bar{\kappa}}_{pqr} - \frac{1}{3} \bar{\kappa}_{qt} e_{rpt} - \frac{1}{3} \bar{\kappa}_{rt} e_{qpt} \right).$$

It should be noted that the condition $\mathbf{g} = \mathbf{0}$ is satisfied in the case of all linear isotropic materials.

Giannakopoulos *et al.* [2] showed that the following reciprocity relationship holds

$$\iint_S \left(P_i u_i^* + Q_i^t \omega_i^{t*} + R \varepsilon^* \right) dS + \sum_\alpha \oint_{C^\alpha} E_i u_i^*\, ds = \iint_S \left(P_i^* u_i + Q_i^{t*} \omega_i^t + R^* \varepsilon \right) dS + \sum_\alpha \oint_{C^\alpha} E_i^* u_i\, ds \tag{5}$$

where $\left(\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\omega}\right)$ and $\left(\mathbf{u}^*, \boldsymbol{\varepsilon}^*, \boldsymbol{\omega}^*\right)$ are elastic states corresponding to generalized loads $\left(\mathbf{P}, \mathbf{Q}^t, R, \mathbf{E}^\alpha\right)$ and $\left(\mathbf{P}^*, \mathbf{Q}^{t*}, R^*, \mathbf{E}^{\alpha*}\right)$. Polyzos *et al.* [8] also proved a reciprocal identity for a specific form of an isotropic, linear elastic solid with microstructure. It is emphasized though that (5) is valid within a more general framework of Mindlin's theory for static analysis of three-dimensional linear elastic solids that accounts for microstructural effects with the only constraint that $g_{ijkpq} = 0$.

Giannakopoulos *et al.* [2] used also the aforementioned reciprocity relationship a methodology similar to that of Mises [7] and Sternberg [9] to show that, if the forces acting on an elastic body are confined to several distinct portions of its surface S_k $\left(k = 1, 2, \ldots\right)$, each lying within a sphere of radius ε, then the displacements at a fixed interior point of the body is of order $O\left(\varepsilon^2\right)$ or smaller if the resultant forces on each S_k are non-zero, $O\left(\varepsilon^3\right)$ or smaller if the corresponding resultant forces vanish on each S_k, and $O\left(\varepsilon^4\right)$ or smaller if the forces are in "astatic" equilibrium (system of parallel forces that remain in equilibrium under an arbitrary change of its direction, the magnitude and sense of forces being maintained) on each S_k.

Applications

As an example, we consider the special case where the elastic strain energy density is of the form $\hat{W}\left(\boldsymbol{\varepsilon}, \hat{\boldsymbol{\kappa}}\right) = \frac{1}{2}\lambda \varepsilon_{ii}\varepsilon_{kk} + \mu \varepsilon_{ij}\varepsilon_{ij} + \frac{\ell^2}{2}\left(\lambda \hat{\kappa}_{ijj}\hat{\kappa}_{ikk} + 2\mu \hat{\kappa}_{ijk}\hat{\kappa}_{ijk}\right)$, where λ and μ are the Lamé constants. The quantity $\hat{\sigma}_{ij} = \hat{\sigma}_{ji} = \bar{\sigma}_{ij} - \hat{\mu}_{kij,k} = \partial \hat{W} / \partial \varepsilon_{ij} - \partial\left(\partial \hat{W} / \partial \hat{\kappa}_{kij}\right) / \partial x_k$ can be written as $\hat{\sigma}_{ij} = \bar{\sigma}_{ij} - \ell^2 \nabla^2 \bar{\sigma}_{ij}$ with $\bar{\sigma}_{ij} = 2\mu \varepsilon_{ij} + \lambda \varepsilon_{kk}\delta_{ij}$, where it was taken into account that $\hat{\kappa}_{ijk} = \varepsilon_{jk,i}$. The corresponding equivalent type III form is

$$\bar{W}\left(\boldsymbol{\varepsilon}, \bar{\boldsymbol{\kappa}}, \bar{\bar{\boldsymbol{\kappa}}}\right) = \frac{\lambda}{2}\varepsilon_{ii}\varepsilon_{kk} + \mu \varepsilon_{ij}\varepsilon_{ij} + \ell^2\left(\frac{\lambda + 3\mu}{9}\bar{\kappa}_{ij}\bar{\kappa}_{ij} - \frac{2\lambda}{9}\bar{\kappa}_{ij}\bar{\kappa}_{ji} + \frac{\lambda}{2}\bar{\bar{\kappa}}_{iij}\bar{\bar{\kappa}}_{kkj} + \bar{\bar{\kappa}}_{ijk}\bar{\bar{\kappa}}_{ijk} + \frac{2\lambda}{3}e_{ijk}\bar{\kappa}_{ij}\bar{\bar{\kappa}}_{kpp}\right).$$

We consider a rectangular block of material with length $2L$ and height H. Figure 1 shows one half of the block. Loads are applied near point A (Fig. 1) and its symmetric point (not shown in the figure) with respect to the x_2-axis; plane strain conditions are considered. The three loading cases $\bar{\mathbf{P}}$ shown in Fig. 1 are analyzed.

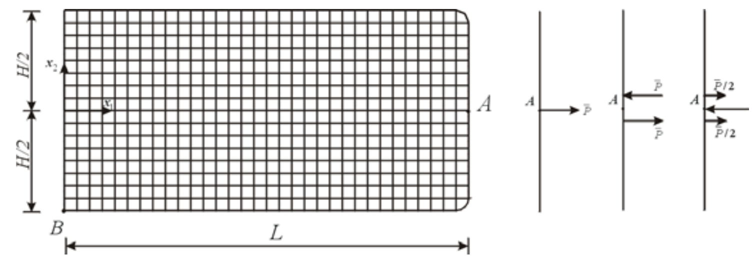

Figure 1

The corresponding problems are solved by using the finite element method. Half of the block is analyzed using a 32×16 finite element mesh (element III9-70 mentioned above). The symmetry conditions $u_1\left(0, x_2\right)=0$ and $\omega_3^{\prime}\left(0, x_2\right)=0$ are used together with the constraint $u_2(0,0)=0$ that eliminates any rigid body motion in the x_2-direction. The distance e that defines the location of the applied loads near point A is equal to one element side, i.e., $e=H / 16$. The first type of load corresponds to non-zero resultant force; in the second case the resultant forces vanishes on S_k; in the third case the forces are in astatic equilibrium on each S_k. The calculations are carried out for $L / H=2$, $\ell / H=0.04$ and $v=0.28$, where v is Poisson's ratio. We define the dimensionless quantity $\hat{v}_B=v_B E /(\bar{P} t)$, where v_B is the vertical displacement of point B, E Young's modulus and t the out-of-plane thickness. The calculated values of $\hat{v}_B$ for the three types of loading are

$$\hat{v}_B^{(1)}=1.64 \times 10^{-1}, \qquad \hat{v}_B^{(2)}=6.03 \times 10^{-2}, \qquad \hat{v}_B^{(3)}=7.68 \times 10^{-9} .$$

Clearly, the following inequalities hold

$$O\left(\hat{v}_B^{(2)}\right) \leq O\left(\varepsilon \hat{v}_B^{(1)}\right), \quad O\left(\hat{v}_B^{(3)}\right) \leq O\left(\varepsilon^2 \hat{v}_B^{(1)}\right), \qquad \varepsilon=e / H=0.0625,$$

thus verifying the aforementioned "Saint-Venant principle".

References

1 Amanatidou, E. and Aravas, N. (2002):"Mixed finite element formulations of strain-gradient elasticity problems", *Comput. Methods Appl. Mech. Engrg.* 191, pp. 1723-1751.

2 Giannakopoulos, A. E., Amanatidou, E. and Aravas, N. to appear (2004): "A reciprocity theorem in linear gradient-elasticity and the corresponding Saint-Venant principle".

3 Koiter, W. T. (1964): "Couple stresses in the theory of elasticity I and II", *Proc. K. Ned. Akad. Wet. (B)* 67, pp. 17-44.

4 Mindlin, R. D. (1964): "Microstructure in linear elasticity", *Archs Ration. Mech. Anal.* 10, pp. 51-78.

5 Mindlin, R. D. and Eshel, N. N. (1968): "On first strain-gradient theories in linear elasticity", *Int. J. Solids Struct.* 4, pp. 109-124.

6 Mindlin, R. D. and Tiersten, H. F. (1962): "Couple-stresses in linear elasticity", *Archs Ration. Mech. Anal.* 11, pp. 415-448.

7 Mises, R. V. (1945): "On Saint-Venant's principle", *Bull. Amer. Math Soc.* 51, pp. 555-562.

8 Polyzos, D., Tsepoura, K. G., Tsiropoulos, S. V. and Beskos, D. E. (2003): "A boundary element method for solving 2-D and 3-D static gradient elastic problems. Part I: Integral formulation", *Comput. Methods Appl. Mech. Engrg.* 192, pp. 2845-2873.

9 Sternberg, E. (1953) : "On Saint-Venant's principle", *Quart. Appl. Math.* 11, pp. 393-402.