Directly Derived Non-Hyper-Singular Boundary Integral Equations I: Petrov-Galerkin Schemes

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Summary

Novel non-hyper-singular boundary-integral-equations for the gradients of the acoustic velocity potential are derived, for solving problems of acoustics governed by the Helmholtz differential equation. Using the basic identities, the strongly singular integral equations for the potential and its gradients are rendered to be only weakly-singular. General Petrov-Galerkin weak-solutions of R-ϕ-BIE, and R-q-BIE are discussed, and Symmetric Galerkin Boundary Element approaches as well.

Introduction

The difficulties in dealing with hyper-singular integrals, and the nonuniqueness, are two of the well known drawbacks of the existing boundary integral equation (BIE) methods for solving acoustic problems, even though the boundary integral equation method offers more advantages over other popular numerical methods such as the finite element method [1]. Burton and Miller [2] developed a combination of the surface Helmholtz integral equation for potential, and the integral equation for the normal derivative of potential at the surface, to circumvent the problem of nonuniqueness at characteristic frequencies. Their method was labeled as CHIE (Composite Helmholtz Integral Equation). The CHIE method, however, introduces the hypersingular integrals, which are computationally costly. Moreover, in CHIE method, the accuracy of the integrations affects the results, and the conventional Gauss quadrature can not be used directly. Regularization techniques are commonly employed by the followers of the CHIE methodology, to improve the approach by reducing the problem to the one involving $O(r^{-1})$ singular integrals near the point of singularity. Chien, Rajiyah, and Atluri [1] employed some known identities of the fundamental solution from the associated interior Laplace problem, to regularize the hypersingular integrals. This concept was used by many successive researchers.

In the present paper, however, novel non-hyper-singular boundary integral equations are derived directly, for the gradients of the velocity potential. The basic idea of using the gradients of the fundamental solution to the Helmholtz differential equation for velocity potential, as *vector test-functions* to write the weak-form of the original Helmholtz differential equation for potential, and thereby directly derive a *non-hyper-singular boundary integral equations for velocity potential gradients*, has its origins in [3,4,5].

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The boundary integral equations for the potential and its gradient, which are used as starting points in the present paper, are only strongly singular [$O(r^{-2})$]. The further regularization of these *strongly singular* ϕ-BIE, and q-BIE, to *only weakly singular* [$O(r^{-1})$] types, which are labeled here as R-ϕ-BIE, R-q-BIE, respectively, is achieved *by using certain basic identities of the fundamental solution of the Helmholtz differential equation for potential*. In addition, in the present paper, we formulate general Petrov-Galerkin methods to solve the R-ϕ-BIE, and R-q-BIE, in their weak senses. By using the test functions in these Petrov-Galerkin schemes to be the energy-conjugates of the respective trial functions, we develop Symmetric Galerkin Boundary Element Methods (SGBEM). We label these SGBEM as SGBEM-R-ϕ-BIE and SGBEM-R-q-BIE, respectively.

Non-hypersingular boundary integral equations

The fundamental solution of the Helmholtz equation is governed by the wave equation,

$$\phi^*_{,ii}(\mathbf{x},\xi) + k^2\phi^*(\mathbf{x},\xi) + \delta(\mathbf{x},\xi) = 0 \tag{1}$$

is well known as the free-space Green's function $\phi^*(\mathbf{x},\xi)$. By using $\overline{\phi}$ as the test function to enforce the Helmholtz equation in terms of the trial function ϕ, in a weak-sense, the weak form of Helmholtz equation can be written as,

$$\int_\Omega \left(\nabla^2\phi + k^2\phi\right)\overline{\phi}\, d\Omega = 0 \tag{2}$$

Through using the fundamental solution $\phi^*(\mathbf{x},\xi)$ as the test function $\overline{\phi}$ in Eq. (2), and with the property from Eq. (1), we obtain the integral equation for ϕ:

$$\phi(\mathbf{x}) = \int_{\partial\Omega} q(\xi)\phi^*(\mathbf{x},\xi)dS - \int_{\partial\Omega}\phi(\xi)\Theta^*(\mathbf{x},\xi)dS \qquad \mathbf{x}\in\Omega \tag{3}$$

where, $q(\xi) = n_k(\xi)\phi_{,k}(\xi)$ and $\Theta^*(\mathbf{x},\xi) = n_k(\xi)\phi^*_{,k}(\mathbf{x},\xi)$ at $\xi\in\partial\Omega$.

Eq. (3) is the conventional BIE for ϕ, which is widely used in literature, and is hereafter referred to as the ϕ-BIE. The nonuniqueness of the Helmholtz integral equation, Eq. (3), is well known; it possesses nontrivial solutions at some characteristic frequencies. If we differentiate Eq. (3) directly with respect to $\mathbf{x}_k$, we obtain the second integral equation for the potential gradients $\phi_{,k}(\mathbf{x})$. One term in this equation is hyper-singular, since $\partial\Theta^*(\mathbf{x},\xi)/\partial x_k$ is of order $O(r^{-3})$ for a 3D problem. A wide body of literature is devoted to deal with the hyper-singularity in this equation.

The novel method in this paper starts *from writing a vector weak-form* [as opposed to a scalar weak-form] of the governing equation Eq. (2) by using the vector test function $\overline{\phi}_{,k}$. By using the gradients of the fundamental solution, viz., $\phi_{,k}^{*}(\mathbf{x},\boldsymbol{\xi})$, as the test functions, we obtain

$$\begin{aligned} &-\phi_{,k}(\mathbf{x})=\int_{\partial\Omega} q(\boldsymbol{\xi})\phi_{,k}^{*}(\mathbf{x},\boldsymbol{\xi})dS \\ &+\int_{\partial\Omega} D_t\phi(\boldsymbol{\xi})e_{ikt}\phi_{,i}^{*}(\mathbf{x},\boldsymbol{\xi})dS+\int_{\partial\Omega} k^2 n_k(\boldsymbol{\xi})\phi(\boldsymbol{\xi})\phi^{*}(\mathbf{x},\boldsymbol{\xi})dS \end{aligned}, \quad D_t=n_r e_{rst}\frac{\partial}{\partial \xi_s} \tag{4}$$

We refer to Eq. (4), hereafter, as the presently proposed non-hyper-singular q-BIE. The ϕ-BIE [Eq. (3)], and q-BIE [Eq. (4)], *are derived independently of each other*. The most interesting feature of the "directly derived" integral equations Eq. (4), for $\phi_{,k}(\mathbf{x})$, *is that they are non-hyper-singular*.

In generally, by using other carefully chosen weak forms of Eq. (1), any number of "properties" of the fundamental solution can be derived [6,7]. *Now, we can use the properties to derive simple, straightforward and elegant further regularizations of the strongly-singular BIEs for ϕ, and $\phi_{,k}$*.Then, One can do the further desingularization of ϕ-BIE and q-BIE easily: Moreover a Petrov-Galerkin scheme can be used to write the weak-form as:

$$\begin{aligned} &\int_{\partial\Omega} w(\mathbf{x})dS_x\int_{\partial\Omega} q(\boldsymbol{\xi})\phi^{*}(\mathbf{x},\boldsymbol{\xi})dS_\xi-\int_{\partial\Omega} w(\mathbf{x})dS_x\int_{\partial\Omega}[\phi(\boldsymbol{\xi})-\phi(\mathbf{x})]\Theta^{*}(\mathbf{x},\boldsymbol{\xi})dS_\xi \\ &=\int_{\partial\Omega} w(\mathbf{x})dS_x\int_{\partial\Omega}^{CPV}\Theta^{*}(\mathbf{x},\boldsymbol{\xi})\phi(\mathbf{x})dS_\xi+\frac{1}{2}\int_{\partial\Omega} w(\mathbf{x})\phi(\mathbf{x})dS_x \end{aligned} \tag{5}$$

where $w(\mathbf{x})$ is a test function on the boundary $\partial\Omega$. If $w(\mathbf{x})$ is chosen as a Dirac delta function, i.e., $w(\mathbf{x})=\delta(\mathbf{x},\mathbf{x}_m)$ at $\partial\Omega$, we obtain the standard "collocation" boundary element method, i.e., BEM-R-ϕ-BIE. If $w(\mathbf{x})$ is chosen to be identical to a function which is energy-conjugate to $\phi(\mathbf{x})$, viz. the trial function $\hat{q}(\mathbf{x})$, we obtain the Symmetric Galerkin "SGBEM-R-ϕ-BIE" form as [6]

Similarly, by using Eq. (4) and the properties, and contracting with $n_k(\mathbf{x})$ on both sides, we can obtain the fully regularized form of Eq. (4), then we can obtain the Petrov-Galerkin scheme as,

$$-\frac{1}{2}\int_{\partial\Omega} q(\mathbf{x})w(\mathbf{x})dS_x = \int_{\partial\Omega} q(\xi)dS_\xi \int_{\partial\Omega}^{CPV} w(\mathbf{x})\hat{\Theta}^*(\mathbf{x},\xi)dS_x$$
$$+\int_{\partial\Omega} D_k w(\mathbf{x})dS_x \int_{\partial\Omega} D_t\phi(\xi)\mathrm{H}_{kt}(\mathbf{x},\xi)dS_\xi \quad (6)$$
$$+\int_{\partial\Omega} w(\mathbf{x})dS_x \int_{\partial\Omega} k^2 n_k(\xi)\phi(\xi)n_k(\mathbf{x})\phi^*(\mathbf{x},\xi)dS_\xi$$

where the kernel function $\hat{\Theta}^*(\mathbf{x},\xi) = -\frac{\partial\phi^*(\mathbf{x},\xi)}{\partial n_x} = n_k(\mathbf{x})\frac{\partial\phi^*(\mathbf{x},\xi)}{\partial\xi_k}$ at $\xi \in \partial\Omega$, and $\mathrm{H}^*_{nt}(\mathbf{x},\xi) = -\delta_{nt}\phi^*(\mathbf{x},\xi)$. If the test function $w(\mathbf{x})$ is chosen to be identical to a function which is energy-conjugate to $q(\mathbf{x})$, viz. the trial function $\hat{\phi}(\mathbf{x})$, we obtain the symmetric Galerkin "SGBEM-R-q-BIE" form.

Numerical results

In the implementation of the "SGBEM-R-ϕ-BIE" and *"SGBEM-R-q-BIE"*, another key step is to evaluate the double area integrals of the weakly singular kernels. As we know, the transformation Jacobian cancels the weak singularity of the kernels. For coincident elements and for elements with common edges or common vertices, the four-dimensional integration domain is divided into several integration subdomains. In each subdomain, a special coordinate transformation is introduced to cancels the singularity.

The sound field radiated by a sphere is studied in this section. The sphere is of unit radius with both a driving surface as well as an admittance surface, which constitute discontinuous boundary conditions. The radiating sphere is studied at the wave number $k = 4.49$ (the second internal eigenvalue of the sphere, and the first eigenvalue is π), which has the largest value of error (~14%) in the numerical solution of CBIE. The exact solution for the radiated field, for the given conditions of Driving surface ($\partial\phi/\partial n = (-0.976 - 0.239i)\cos\theta$), and admittance surface ($\partial\phi/\partial n = (-1.05 + 4.28i)\phi$), is given by $\phi = 0.228i\cos\theta$ on the surface, and $\phi = (-0.00867 + 0.00498i)\cos\theta$ at the far field ($kr = 100$). A comparison between the conventional collocation-based boundary integral equations (ϕ-BIE) viz., the BEM-R-ϕ-BIE approach, the present SGBEM-R-ϕ-BIE and SGBEM-R-q-BIE, and the exact solutions, for the amplitude $|\phi|$ of the velocity potential on the surface and at the far field of the sphere is presented in Fig. 1. The results also show the very high accuracy at the characteristic frequencies in comparison to the conventional boundary element method.

The acoustic scattering of plane waves with unit amplitude (e^{-ikx}) at normal incidence on a rigid cube with length a (a = 1) is considered as the second example to check the practicality of the present method for non-smooth boundaries. The cube is rotated so that the plane waves are toward its corner. For comparison purposes, 3 different-sized models are used for *ka* = 1 (24 elements; 96 elements; 384 elements). The solution at the horizontal plane of symmetry, which is aligned with the incoming wave, is studied. The non-dimensionalized scattered pressure p_s / p_i , at distance *r* from the center of the cube, versus the polar angle is plotted in Fig. 3 for wave number *ka* = 1.0. The solution shows that the method converges, as the number of elements increases.

Closure

The symmetric Galerkin Boundary Element formulations of the regularized forms of newly derived non-hyper-singular boundary integral equations have been presented, in order to overcome the difficulties with hyper-singular integrals. There is no requirement of smoothness of the chosen trial functions for ϕ and *q*, and C^0 continuity is sufficient for numerical implementation. Another advantage of symmetric Galerkin formulation is the symmetry of system matrix, which benefits the solving of the large system equations a lot.

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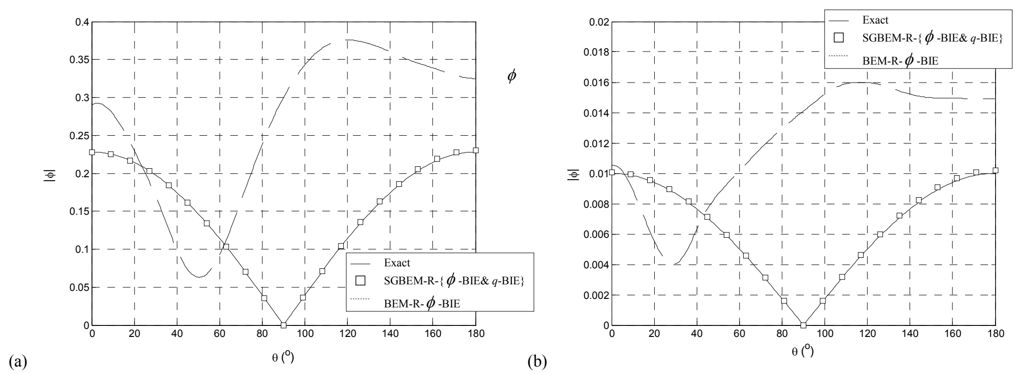

Fig. 1 Solutions of $|\phi|$: (a) on the surface and (b) at the far field (*kr*=100)

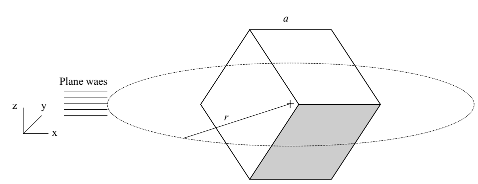

Fig. 2 The geometry and the location of the cube

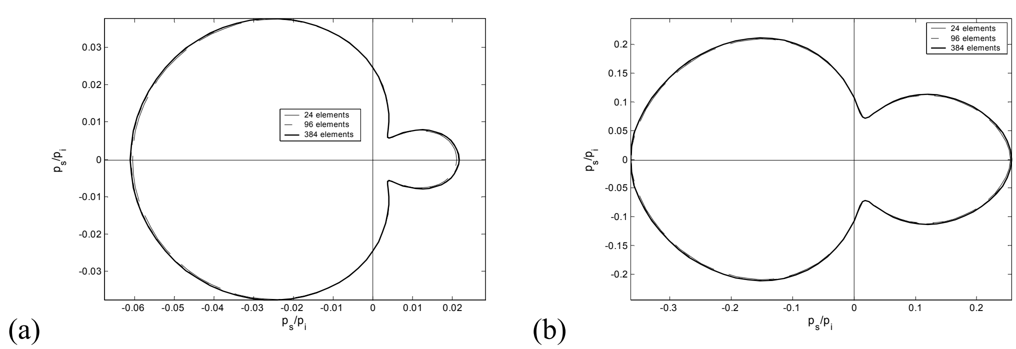

Fig. 3 The angular dependence of ϕ_s / ϕ_i for a cube with (a) r = 1.0; (b) r = 5.0