Contact Mechanics and BEM

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Summary

Boundary Elements Lagrangian formulation to solve contact mechanic or rolling problems is presented showing the basic equations and the main goals on its development.

Introduction

Contact mechanic problems have traditionally occupied a prominent place in the technical engineering literature as the chief result of the need to develop criteria for the design of many elements used in mechanical or civil engineering. The interest on these applications has fostered publications of research or applied works and experimental studies. In the past, most of them were developed to solve specific problems using approximations that are strongly dependent on the case concerned. This lack of generality was mainly due to the problem complexity. In fact, this problem is not simple and implies the solution of highly non-linear equations and a careful examination of the results obtained.

Actually, the quick improvement of computers and numerical techniques as the Finite Element Method (FEM), the Boundary Element Method (BEM) and the Mathematical Programming Techniques (MPT), allows to develop new solution methods to solve the problem. But these methods are far to be simple or uncomplicated, usually they are sophisticated and, to be developed, they require researchers with an important mechanical and numerical background.

Contact mechanic problems are present in the FEM and BEM literature from the beginning. Solution techniques have come a long way since trial and error solution techniques were used. Today, FEM is the most used numerical method by engineers to solve mechanical and structural problems, but BEM is also a valid and competitive alternative to solve some of them, being contact mechanics one of these problems. Having in mind this idea, in this work we present Lagrangian formulations that can be used to solve contact mechanic problems with BEM. Most of these formulations have been implemented by the authors to solve 2D rolling and contact mechanic problems.

Boundary Element equations

The discretization process and the application of BEM procedures, done separately to the two 2D bodies in contact, supply the elastic equations of each body: $\mathbf{H}^{\alpha}\mathbf{u}^{\alpha} = \mathbf{G}^{\alpha}\mathbf{p}^{\alpha}$; $\alpha = A, B$. Grouping these equations, condensing all the variables not associated with the contact areas, introducing the normal distance $\delta_n = \delta_{n0} - \left(u_n^A + u_n^B\right)$, and the static slip velocity $s_t = \xi_t + \theta_t \frac{d}{dx_t}\left(u_t^A - u_t^B\right)$ (doing the spatial derivative using a forward finite differences scheme), [5] and [6], permit to write the BEM elastic equations of the coupled problem as

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$$\delta_n = d_n + S_{nn} p_n + S_{nt} p_t \quad ; \quad s_t = \xi_t + B_{nn} p_n + B_{nt} p_t \tag{1}$$

from a flexibility point of view, or, inverting the equations, from the stiffness point of view

$$p_n = q_n + R_{nn} \delta_n + R_{nt} s_t \quad ; \quad p_t = q_t + R_{tn} \delta_n + R_{tt} p_t \tag{2}$$

For the Normal problem (known tangential tractions $p_t^\alpha = \overline{p_t^\alpha}$) the simplified equations are

$$\delta_n = \overline{d_n} + S_{nn} p_n \quad ; \quad p_n = \overline{q_n} + \overline{R}_{nn} \delta_n \tag{3}$$

while for the Tangential problem (known normal tractions $p_n^\alpha = \overline{p_n^\alpha} = -g/\mu$) are

$$s_t = \overline{\xi_t} + B_{nt} p_t \quad ; \quad p_t = \overline{q_t} + \overline{R}_{tt} p_t \tag{4}$$

Contact Mechanics Lagrangian formulation for BEM

Many different numerical procedures have been used to solve contact mechanic problems from a BEM perspective, mainly trial-and-error methods, load scaling (waiting for single contact state-changes), or mathematical programming techniques as: Sequential linear programming, bilinear approach, quadratic programming, or complementary problems.

After trying many of these techniques to solve contact problems, [3] , [4], [5] and [6], called our attention the work of Alart and Curnier [1], who developed a mixed penalty-duality formulation of the frictional contact problem inspired by the Augmented Lagrangian method to treat its multivalued aspects. They derived an unsymmetrical operator using a quasi-Augmented Lagrangian formulation, showed its properties, and established a necessary and sufficient condition on the friction coefficient that guarantee the uniqueness for the solution of curved, discrete, small slip contacts, and the convergence of Newton's method. Inspired by that work, Christensen [2] proposes a variation of it, giving place to a system of B-differentiable equations.

The cited works were related with the Finite Element Method, but contained general concepts and useful points of view that could be generalized to solve contact problems using the Boundary Element Method equations, most of them related with Lagrangian formulations.

I) Lagrangian formulation of the Normal problem

The normal problem can be formulated as the minimization of a functional $L(\delta_n, p_n, \omega)$ without constraints,

$$\text{Min. } L(\delta_n, p_n, \omega) \quad ; \quad L(\delta_n, p_n, \omega) = \Pi^n(\delta_n) + \sum_{i=1}^{n_p} p_{n_i} \left(\delta_{n_i} - \omega_i^2\right) \tag{5}$$

where $\Pi^n(\delta_n)$ is a function of displacements, ω_i are the separation variables, and n_p the number of contact points.

The first order optimum conditions will be

$$\left.\begin{array}{l}\nabla_{\delta_n} L(\delta_n^*, p_n^*, \omega^*) \equiv \nabla_{\delta_n} \Pi^n(\delta_n) + p_n = 0 \\ \nabla_{p_{n_i}} L(\delta_n^*, p_n^*, \omega^*) \equiv \delta_{n_i} - \omega_i^2 = 0 \quad (i = 1 \ldots n_p) \\ \nabla_{\omega_i} L(\delta_n^*, p_n^*, \omega^*) \equiv \omega_i^* p_{n_i}^* = 0 \quad (i = 1 \ldots n_p)\end{array}\right. \Rightarrow \left\{\begin{array}{c} -(\bar{q}_n + \bar{R}_{nn}\delta_n) + p_n = 0 \\ \delta_n \geq 0 \\ \delta_n^T p_n = 0 \end{array}\right\} \quad (6)$$

where in the first equations in (6), $\nabla_{\delta_n}\Pi^n(\delta_n) + p_n = 0$, have been introduced the BEM condensed elastic equations, $p_n - (\bar{q}_n + \bar{R}_{nn}\delta_n) = 0$ substituting $\nabla_{\delta_n}\Pi^n(\delta_n)$, because in BEM there is not a functional $\Pi^n(\delta_n)$, like the potential energy function, to obtain the elastic equations. Then, the role of elastic equations derived from the functional are occupied by the BEM elastic equations, assuming tha exists a functional from which the BEM equations could be obtained.

The flexibility formulation can be easily obtained operating with the elastic equations.

II) Augmented Lagrangian formulation of the Normal problem

Using (5), it can be defined the Augmented Lagrangian function adding a new term: $r\left(\delta_{n_i} - \omega_i^2\right)^2/2 \quad (r > 0)$, and doing a reorganization

$$\widetilde{L}(\delta_n, p_n, \omega) = \Pi^n(\delta_n) + \frac{1}{2r}\sum_{i=1}^{n_p}\left\{[p_{n_i} + r\left(\delta_{n_i} - \omega_i^2\right)]^2 - p_{n_i}^2\right\} \quad (7)$$

being the first order optimum conditions

$$\begin{array}{ll}\nabla_{\delta_n}\widetilde{L}(\delta_n,p_n,\omega) \equiv \nabla_{\delta_n}\Pi_n(\delta_n) + \left[p_n + r\left(\delta_n - \omega^2\right)\right] = 0 & \\ \nabla_{p_{n_i}}\widetilde{L}(\delta_n, p_n, \omega) \equiv \delta_{n_i} - \omega_i^2 = 0 & (i = 1 \ldots n_p) \\ \nabla_{\omega_i}\widetilde{L}(\delta_n, p_n, \omega) \equiv \omega_i\left[p_{ni} + r\left(\delta_{ni} - \omega_i^2\right)\right] = 0 & (i = 1 \ldots n_p)\end{array} \quad (8)$$

where $\widetilde{L}(\delta_n, p_n, \omega) = L(\delta_n, p_n, \omega)$ for the optimum solution $(\delta_n^*, p_n^*, \omega^*)$.

The equations system (8) is identical to the corresponding (6) knowing that $\delta_{n_i} - \omega_i^2 = 0$. From the last equation of (8) is also possible to define ω_i^2 as $\omega_i^2 = \max(0, \delta_{n_i} + p_{n_i}/r)$, and then $p_{n_i} + r\left(\delta_{n_i} - \omega_i^2\right) = \min(0, p_{n_i} + r\delta_{n_i})$, which permits a new definition of $\widetilde{L}$

$$\widetilde{L}(\delta_n, p_n) = \Pi^n(\delta_n) + \frac{1}{2r}\sum_{i=1}^{n_p}\left[\min(0, p_{n_i} + r\delta_{n_i})^2 - p_{n_i}^2\right] \quad (9)$$

If the definitions of positive and negative euclidean distance are introduced in (9) ($D_{\mathbf{R}^+}(x) = -\min(0,x)$, and $D_{\mathbf{R}^-}(x) = \max(0,x)$, both related by $D_{\mathbf{R}^+}(x)^2 = x^2 - D_{\mathbf{R}^-}(x)^2$), then

$$\widetilde{L}(\delta_n, p_n) = \Pi^n(\delta_n) + \frac{1}{2r}\sum_{i=1}^{n_p}[2rp_{n_i}\delta_{n_i} + r^2\delta_{n_i}^2 - D_{R^-}(p_{n_i} + r\delta_{n_i})^2] \quad (10)$$

being now the first order optimum conditions

$$\left.\begin{array}{l}\nabla_{\delta_n}\widetilde{L}(\delta_n, p_n) \equiv \nabla_{\delta_n}\Pi^n(\delta_n) + \min(0, p_n + r\delta_n) = 0 \\ \nabla_{\mathbf{p}_n}\widetilde{L}(\delta_n, p_n) \equiv -\frac{1}{r}\left[p_n - \min(0, p_n + r\delta_n)\right] = 0\end{array}\right. \Rightarrow \left\{\begin{array}{c} -(\bar{q}_n + \bar{R}_{nn}\delta_n) + p_n = 0 \\ p_n - \min(0, p_n + r\delta_n) = 0\end{array}\right\} \quad (11)$$

equations that from the BEM point of view, can be identified with the condensed flexibility representation of contact problems. As in Lagrangian formulation, the stiffness representation can be formulated inverting the equilibrium equations.

III) Lagrangian formulation of the Tangential problem

For pure friction problems, known normal tractions, it is not possible to formulate equations as in (5), because the friction law implicates constraints of a different nature. This inconvenient is bypassed writing a dual Lagrangian formulation, analogous to (5) but changing the constraints to the dual variables, obtaining

$$\underset{s_t}{\text{Min}} \left\{ \underset{p_t \in C_g}{\text{Max}}\, L(s_t, p_t) \right\} \quad ; \quad L(s_t, p_t) = \Pi^t(s_t) - \sum_{i=1}^{n_p} p_{t_i} s_{t_i} \tag{12}$$

where C_g is the feasible set of tangential tractions for the known normal tractions.

IV) Augmented Lagrangian formulation of the Tangential problem

For the tangential problem, the only valid expression for the Augmented Lagrangian is the one obtained from equation (10) by analogy, i.e., changing the normal variables by the tangential ones, and defining the euclidean distance not in R^- but in C_g, $D_{\mathbb{C}_g}(x) = \max(x-g,0) - \min(x+g,0)$,

$$\widetilde{L}(s_t, p_t) = \Pi^t(s_t) + \frac{1}{2r} \sum_{i=1}^{n_p} \left[2r p_{t_i} s_{t_i} - r^2 s_{t_i}^2 + D_{\mathbb{C}_{g_i}}(p_{t_i} - r s_{t_i})^2 \right] \tag{13}$$

being the sign changes due to the dissipation condition of the friction law.

Using now the projection function $P_{\mathbb{C}_g}(x) = x - \text{sgn}(x) D_{\mathbb{C}_g}(x)$, and its relation with the derivative of distance function, $\frac{d}{dx}[D_{\mathbb{C}_g}(x)^2] = 2[x - P_{\mathbb{C}_g}(x)]$, it is possible to obtain the optimum conditions as

$$\begin{array}{l} \nabla_{\mathbf{s}_t} \widetilde{L}(s_t, p_t) \equiv \nabla_{\mathbf{s}_t} \Pi^t(s_t) + P_{\mathbb{C}_\mathbf{g}}(p_t - r s_t) = 0 \\ \nabla_{\mathbf{p}_t} \widetilde{L}(s_t, p_t) \equiv \frac{1}{r} \left[p_t - P_{\mathbb{C}_\mathbf{g}}(p_t - r s_t) \right] = 0 \end{array} \Rightarrow \left\{ \begin{array}{c} -(\overline{q}_t + \overline{R}_{tt} p_t) + p_t = 0 \\ p_t - P_{\mathbb{C}_\mathbf{g}}(p_t - r s_t) = 0 \end{array} \right\} \tag{14}$$

V) Lagrangian formulation of the Coupled problem

Due to the coupling between normal and tangential variables, the general contact problem becomes an unconventional optimization problem, being recommendable to use "quasi" as prefix in the formulation name. From equations (5) and (12) the quasi-Lagrangian dual formulation could be written as

$$\underset{\delta_n}{\text{Min}} \left\{ \underset{p_n \le 0}{\text{Max}} \left\{ \underset{p_t \in C_{\mu \mathbf{p} n}}{\text{Max}}\, L(\delta_n, s_t, p_n, p_t) \right\} \right\} \tag{15}$$

being

$$L(\delta_n, s_t, p_n, p_t) = \Pi(\delta_n, s_t) + \sum_{i=1}^{n_p} (p_{n_i} \delta_{n_i} - p_{t_i} s_{t_i}) \tag{16}$$

where the unconventionality comes from the unknown normal tractions in $C_{\mu\mathbf{p}_n}$.

VI) Augmented Lagrangian formulation of the Coupled problem

Combining the previous formulations of normal and tangential problems (9) and (13), it can be obtained the Augmented Lagrangian formulation of the coupled problem

$$\begin{aligned}\widetilde{L}(\delta_n, \mathbf{s}_t, \mathbf{p}_n, \mathbf{p}_t) \quad = \quad & \Pi(\delta_n, \mathbf{s}_t) + \frac{1}{2r}\sum_{i=1}^{n_p}[\min(0, p_{n_i} + r\delta_{n_i})^2 - p_{n_i}^2 + \\ & + 2rp_{t_i}s_{t_i} - r^2 s_{t_i}^2 - \mathbb{D}_{\mathbb{C}_{-\mu\min(0,p_{n_i}+r\delta_{n_i})}}(p_{t_i} - rs_{t_i},)^2]\end{aligned} \tag{17}$$

where the distance to the friction limit $\mu\min(0, p_{n_i} + r\delta_{n_i})$ becomes the projection function on a valid region of p_n. The optimum conditions of this function are

$$\left\{\begin{array}{c} -(\overline{q}_n + \overline{R}_{nn}\delta_n) + p_n \\ -(\overline{q}_t + \overline{R}_{tt}p_t) + p_t \\ p_n - \min(0, p_n + r\delta_n) \\ p_t - P_{\mathbb{C}_\mathbf{g}}(p_t - rs_t) \end{array}\right\} = 0 \tag{18}$$

This equation system has been widely used by the authors to solve 2D contact and rolling problems, [5] and [6].

Solution procedure

The equations obtained by the Lagrangian formulation are B-differentiable, the concept of B-differentiability is related to the non-linearity of the directional derivative, which means that an specialized method, as the Generalized Newton's Method with line search (*GNMls*), should be used to solve the problem. *GNMls* is an extension of the Newton's Method for B-differentiable functions formulated by Pang [7].

The *GNMls* algorithm for solving the non-linear equation $G(z) = 0$, being $G(z)$ a B-differentiable function, is formulated from the generalized first order expansion

$$G(z^k) + BG(z^k)\eta_\mathbf{z}^k = 0 \tag{19}$$

where $BG(z^k)$ means B-derivative.

Sometimes the solution could involve a big computational effort because the non-linearity of $BG(z^k)\eta_\mathbf{z}^k$. In order to reduce this effort some previous researchers solved accurately contact problems avoiding the points where the non-linearity could happen or neglecting the non-linear part of the derivative on these points. In contact problems the points to avoid are those where the inequalities formulated during the iterative solution process change to equalities. These points, lines or surfaces, are the frontiers between zones where the functions have different slopes and then, their derivatives do not have the same value for all the directions. During the iterative process it is very unusual to have a trying solution z^k exactly over one of these points, this is the reason because the convergence is not usually affected if those points are avoided, or some part of their derivative neglected. The non-linear part of the generalized Newton's equations, is then linearized and computed as:

$$BG(z^k)\eta_{\mathbf{z}}^k = \left(\bigtriangledown G^{LD} + \partial G^{NLD}\right)\eta_{\mathbf{z}}^k \quad (20)$$

where $\bigtriangledown G^{LD}$ is the jacobian of the linear part, and ∂G^{NLD} is a pseudo-jacobian of the non-linear part, computed avoiding the non-linear points.

Conclusions

The Boundary Elements Lagrangian formulation to solve 2D contact mechanic or rolling problems have been presented showing the basic equations and the main goals on their development. The formulation can be easily written from the point of view of stiffness or flexibility.

Acknowledgement

This work was funded by Spain's Ministerio de Ciencia y Tecnología, in the framework of Projects DPI2000-1642 and DPI2003-00487.

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