Transient analysis of buried dynamic sources in an anisotropic elastic half-space

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Summary

A method to deal with the two-dimensional transient problem of a line force or dislocation in an anisotropic elastic half-space is developed in this investigation. The proposed formulation is similar to Stroh's formalism for anisotropic elastostatics in that the two-dimensional anisotropic elastodynamic problem is cast into a six-dimensional eigenvalue problem and the solution is expressed in terms of the eigenvalues and eigenvectors. An analytic solution is obtained without performing integral transforms. A numerical example is presented for a silicon half-space subjected to a line force.

Introduction

The Stroh formalism is widely recognized as an elegant and powerful method for two-dimensional anisotropic elastostatics. A distinctive feature of the Stroh formalism is that the general solution is provided in terms of the eigenvalues and eigenvectors of a six-dimensional eigenvalue problem. The general solution contains three arbitrary complex functions. The functions can often be found by taking advantage of the orthogonality relations among the eigenvectors in conjunction with theories of analytic functions. The readers are referred to [1] for more detailed discussions.

Elastodynamic problems are usually studied by integral transform methods. In these methods a formal solution in the transform space is first obtained. Considerable effort is then involved in the inversion from the transform space to the physical space. For isotropic media, the Smirnov-Soloblev method [2] is an alternative method to solve self-similar elastodynamic problems without the need of integral transforms. The Smirnov-Soloblev method has been extended to general anisotropic elastic materials by Wu [3]. The formulation by Wu is similar to Stroh's formalism and preserves many of its advantages.

Self-similar problems are problems for which the displacements are homogeneous functions of time t and position $\mathbf{x}$. The corresponding physical systems thus involve neither a characteristic length nor a characteristic time. In the present problem of a buried source in a half-space the depth of the source appears as a characteristic length. It is therefore non-self-similar and the formulation developed by Wu [3] cannot be applied. In this paper we modify Wu's formulation to treat the problem of interest. The present

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formulation also casts the two-dimensional anisotropic elastodynamic problem into a six-dimensional eigenvalue problem and the general solution is directly expressed in terms of the eigenvalues and eigenvectors in the time domain.

Formulation

The equations of motion expressed in terms of displacements are

$$\mathbf{Q}\mathbf{u}_{,11}+(\mathbf{R}+\mathbf{R}^T)\mathbf{u}_{,12}+\mathbf{T}\mathbf{u}_{,22}=\rho\ddot{\mathbf{u}} \tag{1}$$

where the matrices $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{T}$ are related to elastic constants C_{ijks} by $Q_{ik}=C_{i1k1}$, $R_{ik}=C_{i1k2}$, $T_{ik}=C_{i2k2}$. Let the displacement be assumed as $\mathbf{u}(x_1,x_2,t)=\mathbf{u}(w)$ with the variable $w(x_1,x_2,t)$ implicitly defined by

$$\phi(w,x_1,x_2,t)=wt-x_1-p(w)x_2-q(w)=0 \tag{2}$$

where $p(w)$ is the function of w stipulated by equation (1) and $q(w)$ is an arbitrary given function of w. The special case $q(w)=0$ has been discussed by Wu [3]. Let $\mathbf{u}'(w)$ be expressed as $\mathbf{u}'(w)=f'(w)\mathbf{a}(w)$ where $f(w)$ is an arbitrary scalar function of w and $f'(w)$ its derivative with respect to w. It follows that $\mathbf{u}(w)$ is a solution of equation (1) if

$$\mathbf{D}(p,w)\mathbf{a}(w)=\mathbf{0} \tag{3}$$

where $\mathbf{D}(p,w)$ is given by $\mathbf{D}(p,w)=\mathbf{Q}+p(\mathbf{R}+\mathbf{R}^T)+p^2\mathbf{T}-\rho w^2\mathbf{I}$. The general solution of equation (1) may be represented as

$$\mathbf{u}(x_1,x_2,t)_{,1}=2\,\mathrm{Re}\{\sum_k f_k'(w_k)(\partial w_k/\partial x_1)\mathbf{a}_k(w_k)\} \tag{4}$$

$$\mathbf{u}(x_1,x_2,t)_{,2}=2\,\mathrm{Re}\{\sum_k p_k(w_k)f_k'(w_k)(\partial w_k/\partial x_1)\mathbf{a}_k(w_k)\} \tag{5}$$

$$\dot{\mathbf{u}}(x_1,x_2,t)=-2\,\mathrm{Re}\{\sum_k w_k f_k'(w_k)(\partial w_k/\partial x_1)\mathbf{a}_k(w_k)\} \tag{6}$$

where f_k is an arbitrary function of w, $k=1,2,3$ or $4,5,6$. The choice of the range of k depends on whether up-going rays or down-going rays are considered. The general solutions of the stress vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ can be expressed as

$$\mathbf{t}_1(x_1,x_2,t)=2\,\mathrm{Re}\{\sum_k f_k'(w_k)[\rho w_k^2(\partial w_k/\partial x_1)\mathbf{a}_k(w_k)-p_k(w_k)(\partial w_k/\partial x_1)\mathbf{b}_k(w_k)]\} \tag{7}$$

$$\mathbf{t}_2(x_1, x_2, t) = 2\,\mathrm{Re}\{\sum_k f_k'(w_k)(\partial w_k / \partial x_1)\mathbf{b}_k(w_k)\} \tag{8}$$

where

$$\mathbf{b}_k(w) = (\mathbf{R}^T + p_k(w)\mathbf{T})\mathbf{a}_k(w) \tag{9}$$

Equation (9) can be cast into the following six-dimensional eigenvalue problem

$$\mathbf{N}(w)\boldsymbol{\xi}(w) = p(w)\boldsymbol{\xi}(w) \tag{10}$$

where $\mathbf{N}(w) = \begin{pmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3(w) & \mathbf{N}_1^T \end{pmatrix}$, $\boldsymbol{\xi}(w) = \begin{pmatrix} \mathbf{a}(w) \\ \mathbf{b}(w) \end{pmatrix}$,

$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^T$, $\mathbf{N}_2 = \mathbf{T}^{-1}$, $\mathbf{N}_3(w) = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q} + \rho w^2 \mathbf{I}$ .Equation (10) is in the same form as that in Stroh's formalism for steady state motion [4]. The p and $\boldsymbol{\xi}$ are the eigenvalue and right eigenvector, respectively, of $\mathbf{N}$.

Buried Dynamic Sources In An Infinite Medium

Consider a line force $\mathbf{F}$ and a dislocation with Burgers vector $\boldsymbol{\beta}$ which appear at $x_1 = 0$ and $x_2 = h$ at time $t = 0$ and stay constant thereafter in an initially stress-free infinite medium. The configuration is shown in Figure 1.

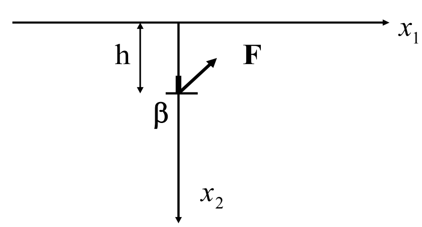

Figure 1. Configuration of the problem of interest.

The associated jump conditions are given by

$$\mathbf{t}_2^+(x_1, h, t) - \mathbf{t}_2^-(x_1, h, t) = -\delta(x_1)H(t)\mathbf{F} \tag{11}$$

$$\mathbf{u}_{,1}^+(x_1, h, t) - \mathbf{u}_{,1}^-(x_1, h, t) = -\delta(x_1)H(t)\boldsymbol{\beta} \tag{12}$$

where δ is the Dirac delta function, H is the Heaviside step function, and superscripts + and – denote the limiting values as $x_2 \to h^+$ and $x_2 \to h^-$, respectively. The solution for the line force has been obtained by Wu [3] and that for the line dislocation may be derived similarly. The result is

$$\dot{\mathbf{u}}^{(0)} = -(1/\pi)\,\mathrm{Im}\{\sum_{k=1}^{3} c_k(w_k)(\partial w_k / \partial x_1)\mathbf{a}_k(w_k)\} \tag{13}$$

$$\mathbf{t}_2^{(0)} = (1/\pi)\,\mathrm{Im}\{\sum_{k=1}^{3} (c_k(w_k)/w_k)(\partial w_k / \partial x_1)\mathbf{b}_k(w_k)\} \tag{14}$$

where $c_k(w_k) = \mathbf{a}_k^T(w_k)\mathbf{F} + \mathbf{b}_k^T(w_k)\boldsymbol{\beta}$, w_k in equation (2) is determined by taking $q(w) = -p(w)h$ so that $w_k = y_1 + p_k(w_k)y_2$ with $y_1 = x_1/t$, $y_2 = (x_2 - h)/t$.

Buried Dynamic Sources In A Half-Space

Consider a line force $\mathbf{F}$ and a dislocation with Burgers vector $\boldsymbol{\beta}$ which appear at $x_1 = 0$ and $x_2 = h$ at time $t = 0$ and stay constant thereafter in the half-space $x_2 > 0$. The conditions on the boundary $x_2 = 0$ is given by

$$\mathbf{t}_2(x_1, t) = \mathbf{0} \tag{15}$$

The resulting $\dot{\mathbf{u}}$ and $\mathbf{t}_2$ may be expressed as $\dot{\mathbf{u}} = \dot{\mathbf{u}}^{(0)} + \dot{\mathbf{u}}^{(1)}$ and $\mathbf{t}_2 = \mathbf{t}_2^{(0)} + \mathbf{t}_2^{(1)}$ where $\dot{\mathbf{u}}^{(0)}$ is the velocity due to the sources in an infinite medium and $\dot{\mathbf{u}}^{(1)}$ is the velocity due to the reflected waves from the free surface. Let

$$w_{jk}t = x_1 + p_j(w_{jk})x_2 - p_k(w_{jk})h \tag{16}$$

Note that at $x_2 = 0$, $w_{jk}(x_1, t) = w_k(x_1, t)$. $\dot{\mathbf{u}}^{(1)}(x_1, x_2, t)$ and $\mathbf{t}_2^{(1)}(x_1, x_2, t)$ can be expressed as

$$\dot{\mathbf{u}}^{(1)} = -(1/\pi)\sum_{k=4}^{6}\sum_{j=1}^{3}\mathrm{Im}\{(\partial w_{jk}/\partial x_1)\mathbf{A}(w_{jk})\mathbf{I}_j\mathbf{B}^{-1}(w_{jk})\hat{\mathbf{B}}(w_{jk})\mathbf{I}_{k-3}(\hat{\mathbf{A}}^T(w_{jk})\mathbf{F} + \hat{\mathbf{B}}^T(w_{jk})\boldsymbol{\beta})\} \tag{17}$$

$$\mathbf{t}_2^{(1)} = (1/\pi)\sum_{k=4}^{6}\sum_{j=1}^{3}\mathrm{Im}\{(1/w_{jk})(\partial w_{jk}/\partial x_1)\mathbf{B}(w_{jk})\mathbf{I}_j\mathbf{B}^{-1}(w_{jk})\hat{\mathbf{B}}(w_{jk})\mathbf{I}_{k-3}(\hat{\mathbf{A}}^T(w_{jk})\mathbf{F} + \hat{\mathbf{B}}^T(w_{jk})\boldsymbol{\beta})\} \tag{18}$$

where $\mathbf{I}_j = \mathbf{e}_j\mathbf{e}_j^T$ ($\mathbf{e}_j$ is the unit vector in the x_j – direction), $\mathbf{A}(w) = [\mathbf{a}_1(w)\quad \mathbf{a}_2(w)\quad \mathbf{a}_3(w)]$, $\mathbf{B}(w) = [\mathbf{b}_1(w)\quad \mathbf{b}_2(w)\quad \mathbf{b}_3(w)]$, $\hat{\mathbf{A}}(w) = [\mathbf{a}_4(w)\quad \mathbf{a}_5(w)\quad \mathbf{a}_6(w)]$, $\hat{\mathbf{B}}(w) = [\mathbf{b}_4(w)\quad \mathbf{b}_5(w)\quad \mathbf{b}_6(w)]$.

Numerical Example

For fixed x_1, x_2 and t expand $\phi(w,t)$ in equation (2) about w_0 up to the second order terms by Taylor's series,

$$\phi(w) = \phi_0 + (\frac{\partial\phi}{\partial w})_0 \Delta w + \frac{1}{2}(\frac{\partial^2\phi}{\partial w^2})_0 (\Delta w)^2 = 0 \tag{19}$$

where $\Delta w = w - w_0$, and $(f)_0 = f(w_0)$. Equation (19) can be regarded as a quadratic equation of Δw:

$$a(\Delta w)^2 + 2b\Delta w + c = 0 \tag{20}$$

In equation (20), a, b, and c are given by $a = -p''_j(w_0)x_2 + p''_k(w_0)h$, $b = t - p'_j(w_0)x_2 + p'_k(w_0)h$, $c = 2\phi_0$. Equation (20) yields $\Delta w = (-b + \sqrt{b^2 - ac})/a$ for a given w_0. Let $w_1 = w_0 + \Delta w$. If $|\phi(w_1)| < \varepsilon$, where ε is a preset error, then w_1 is accepted as the solution of w for given x_1, x_2 and t. Otherwise Equation (20) is solved again with w_0 replaced by w_1. The process is repeated until the error criterion is met.

The vertical displacement due to a unit vertical line impulse has been calculated for silicon. The result is expressed in the following dimensionless form:

$$(V_0)_y = [(t\pi C_{44})/\tau](\mathbf{G}_{0f})_{22},\ (V_1)_y = (V_0)_y + [(t\pi C_{44})/\tau](\mathbf{G}_{1f})_{22} \tag{21}$$

where $\mathbf{G}_{0f}$ and $\mathbf{G}_{1f}$ are given by

$$\mathbf{G}_{0f}(x_1, x_2, t) = -(1/\pi)\,\mathrm{Im}\{\sum_{k=1}^{3}(\partial w_k / \partial x_1)\mathbf{a}_k(w_k)\mathbf{a}_k^T(w_k)\} \tag{22}$$

$$\mathbf{G}_{1f}(x_1, x_2, t) = -(1/\pi)\sum_{k=4}^{6}\sum_{j=1}^{3}\mathrm{Im}\{(\partial w_{jk} / \partial x_1)\mathbf{A}(w_{jk})\mathbf{I}_j\mathbf{B}^{-1}(w_{jk})\hat{\mathbf{B}}(w_{jk})\mathbf{I}_{k-3}\hat{\mathbf{A}}^T(w_{jk})\} \tag{23}$$

$\tau = tc_0/h$, $c_0 = \sqrt{C_{44}/\rho}$, and ρ is density. The elastic constants of silicon used for calculations were C_{11}=165Gpa, C_{12}=63Gpa and C_{44}=79Gpa. The (x_1,x_3)-plane is on the (111) surface and the x_1-axis is in the $[1\bar{1}0]$ direction. The result of $(V_1)_y$ as a function of τ for $x_1/h = 1.6$ and $x_2/h = 1.2$ is given in Figure 2. It is seen that all wave arrivals are accurately captured.

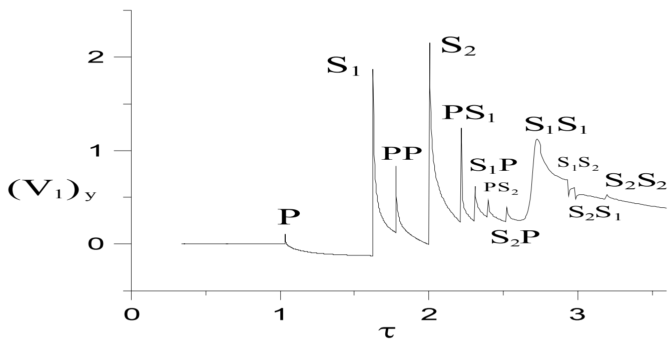

Figure 2 Dimensionless vertical displacement $(V_1)_y$ for $x_1 / h = 1.6$ and $x_2 / h = 1.2$

Conclusion

The formulation developed by Wu [3] is generalized to treat buried dynamic sources in an anisotropic elastic half-space. The displacement or traction fields in time domain have been obtained without using integral transforms. The numerical result shows that the dynamic responses can be efficiently calculated and the complicated wave phenomena accurately captured.

Reference

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