Interconnection between spatial and temporal instability in a forced mixing layer

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Summary

The problem of spatial instability of a parametrically excited stratified mixing layer is considered together with the corresponding temporal instability problem. A relatively simple iteration procedure yielding the solution of the spatial stability problem is developed. It is shown that the eigenvector of the first iteration of the above procedure coincides with the eigenvector of the temporal stability problem, which gives one a possibility to perform a simple and robust comparison of the two different instabilities.

Introduction

Most of computational modeling of instabilities developing in mixing layers is being performed in so-called temporal formulation, in which the spatial flow periodicity is defined and temporal evolution of the perturbations is studied. On the other hand, most of experiments, as well as mixing layers appearing in the nature and technical applications relate to so-called spatial instability. Here two fluid flows meet at a certain point and the instability develops in space, in the direction of mean velocity. The stability analysis for the spatially growing perturbations results in non-linear eigenvalue problem for the complex spatial wavenumber (spatial increment). The temporal problem, however, can be reduced to a linear eigenproblem (e.g., the Orr-Sommerfeld equation), which can be treated by standard means of stability analysis. It is not clear, however, how the results of the spatial and temporal stability analysis can be compared and to which extent they are different.

In the present study we develop a relatively simple iteration procedure, which yields the solution of the spatial stability problem. It occurs that the first iteration of this procedure yields the solution of the temporal problem. Thus, both problems can be compared just by comparison of results obtained at the first and the last iterations of the same iteration procedure.

Formulation of spatial and temporal problems

Consider flow of a Boussinesq incompressible fluid in a thermally stratified mixing layer. The flow is described by the momentum, energy and continuity equations

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$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{v} + g\beta\left(T - \overline{T}\right)e_y$$
$$\nabla\cdot\mathbf{v} = 0, \quad \frac{\partial T}{\partial t} + (\mathbf{v}\cdot\nabla)T = \kappa\Delta\mathbf{v} \tag{1}$$

The layout of the *spatial problem* is defined in accordance with the mixing layer experiments, e.g. [1]. Two layers of air having different temperatures T_1 and T_2 and moving with different horizontal velocities U_1 and U_2 meet at the end of a splitter plate. A flapper installed at the end of the splitter plate introduces the perturbation of a constant circular frequency ω_0, which defines further development of the instability. We consider this problem in a rectangular domain $0 \le x \le X$, $-H \le y \le H$. The boundary conditions are

at $y = \pm H$: $$v_y = 0, \quad \frac{\partial v_x}{\partial y} = 0, \quad \frac{\partial T}{\partial y} = 0 \tag{2}$$

at $x = 0$: $$v_y = 0, \quad v_x = U_0(y) = U_1 + \frac{1}{2}(U_2 - U_1)\left[1 + f\left(\frac{y}{\delta_v}\right)\right], \tag{3}$$

$$T = T_0(y) = T_1 + \frac{1}{2}(T_2 - T_1)\left[1 + g\left(\frac{y}{\delta_T}\right)\right] \quad , \tag{4}$$

where functions $f(z)$ and $g(z)$ describe initial velocity and temperature profiles, e.g., $f(z) = g(z) = \tanh(z)$, and δ_v and δ_T are, respectively, thicknesses of the velocity and the density layers. For the needs of computational modeling the non-reflecting boundary conditions are posed on the outer boundary $x = X$. The initial state is defined by Eqs. (3) and (4) and an additional equation in a single grid point

$$v_y(x = 0, y = 0) = a\sin(\omega_0 t), \tag{5}$$

with small amplitude a and given frequency ω_0. The condition (5) introduces the monochromatic perturbation in the flow and therefore models the flapper.

To render the equations dimensionless we introduce the following scales, respectively for the length, time, velocity, pressure and temperature: $[L] = (U_1 + U_2)/2\omega_0$, $[t] = L/(U_2 - U_1)$, $[v] = U_2 - U_1$, $[p] = \rho(U_2 - U_1)^2$, and $[T] = T_2 - T_1$. The mean values of the velocity and the temperature are defined as $\overline{U} = (U_1 + U_2)/2$ and $\overline{T} = (T_1 + T_2)/2$. The scales are chosen in such a way that the dimensionless length over which the perturbation (5) is advected by the mean velocity $\overline{U}$ during one period of oscillations $\tau = 2\pi/\omega_0$ is $x_\tau = \overline{U}\tau/[L] = 2\pi$. This yields the spatial period of the mean flow (i.e., before the instability sets in) $\alpha_0 = 2\pi/x_\tau = 1$. The equations (1) in the dimensionless form become

$$
\begin{gathered}
\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\nabla p+\frac{1}{Re}\Delta\mathbf{v}+Ri\left(T-\overline{T}\right)\mathbf{e}_y \\
\nabla\cdot\mathbf{v}=0,\quad \frac{\partial T}{\partial t}+(\mathbf{v}\cdot\nabla)T=\frac{1}{Pe}\Delta T
\end{gathered}
\tag{6}
$$

where $Re=[v][L]/\nu$ is the Reynolds number, $Pe=[v][L]/\kappa$ is the Peclet number and $Ri=g\beta[T][L]/(U_2-U_1)^2$ is the Richardson number. In the stability studies for the *spatial problem* we assume that the perturbations behave as $\sim\exp[i(\alpha x+\omega_0 t)]$ and look for the complex spatial wavenumber α_s, which yields the fastest spatial growth of the perturbation having the fixed temporal frequency ω_0.

For the *temporal problem* we consider two layers of air moving at the opposite velocities U_{max} and $-U_{max}$. This problem is also described by equations (1), but is considered in a the shorter rectangular domain $0\le x\le X_t=2\pi/\alpha_t$, $-H\le y\le H$, where α_t is the fixed spatial period of the temporal problem. The boundary conditions for the temporal problem are:

at $y=-H$: $\quad v_y=0,\quad v_x=-U_{\max},\quad T=T_1\quad,$ (7)

at $y=H$: $\quad v_y=0,\quad v_x=U_{\max},\quad T=T_2\quad,$ (8)

and periodicity conditions in the horizontal direction:

$$\mathbf{v}(x=0)=\mathbf{v}(x=X_t),\quad p(x=0)=p(x=X_t),\quad T(x=0)=T(x=X_t) \tag{9}$$

Initial velocity and temperature profiles are defined at $t=0$:

$$
\begin{gathered}
v_y(t=0)=0,\quad v_x(t=0)=U_{\max}f\left(\frac{y}{\delta_v}\right), \\
T(t=0)=T_0(y)=T_1+\frac{1}{2}(T_2-T_1)\left[1+g\left(\frac{y}{\delta_T}\right)\right]
\end{gathered}
\tag{10}
$$

where the functions $f(z)$ and $g(z)$ are the same as for spatial problem.

To render the *temporal problem* dimensionless the following scales of length, time, velocity, pressure and temperature are chosen: $[L]=\alpha_t^{-1}$, $[t]=(\alpha_t U_{\max})^{-1}$, $[v]=U_{\max}$, $[p]=\rho U_{\max}^2$, and $[T]=T_2-T_1$. Obviously, the dimensionless temporal problem is also described by the equations (6). Its dimensionless horizontal coordinate varies from zero to 2π, so that its dimensionless spatial period is unity.

We are interested in the following comparison. Assume that velocities of the spatial and temporal problems are connected as $U_{\max}=(U_2-U_1)/2$, and all other parameters are equal. Assume also, that the spatial instability takes place with the dimensionless spatial increment α_s, which is a complex number slightly diverging from the real unity. We observe the development of the instability (e.g., growth of a Kelvin-Helmholz billow) in the reference frame moving with the mean velocity $\overline{U}$ and compare it with the instability developing in the temporal problem. How different will be the spatial and temporal instabilities? An indication of this difference will be the distance between α_s and unity.

In the following we describe a simple iterative procedure which yields both the spatial and the temporal fastest growing perturbations, and therefore the way to compare them. We assume that the basic flow is given by the same dimensionless temperature profile and by the dimensionless velocity profiles $U_s(y)$ and $U_t(y)$ for the spatial and the temporal problems, respectively:

$$T(y)=\frac{1}{2}\left[1+g\left(\frac{y}{\delta_T}\right)\right],\quad U_s(y)=\frac{1}{2}\left[1+f\left(\frac{y}{\delta_V}\right)\right],\quad U_t(y)=f\left(\frac{y}{\delta_V}\right) \tag{11}$$

Here the temperature and the spatial velocity are rendered dimensionless by the transformations $T\rightarrow(T-T_1)/(T_2-T_1)$ and $U\rightarrow(U-U_1)/(U_2-U_1)$. The perturbations are defined as $A(y)\exp[i(\alpha x+\omega t)]$, where A(y) is the amplitude, α and ω are the spatial and the temporal generalized frequencies. For the spatial problem we assume that α is complex and ω is a given real, while for the temporal problem α is real (according to our scaling $\alpha = 1$) and ω is complex. Note, that according to the scales chosen for the *spatial problem*:

$$\omega=\omega_0\cdot[t]=\frac{1}{2}\frac{U_1+U_2}{U_1-U_2}=\frac{1}{2\lambda},\ \text{where}\ \lambda=\frac{U_1-U_2}{U_1+U_2} \tag{12}$$

It is emphasized that (12) holds only for the spatial case. The new parameter λ is zero when $U_1 = U_2$ (no shear) and is infinite for the temporal case where $U_2 - U_1 = 2U_{max}$.

The linearized stability problem for both spatial and temporal cases is described by the non-isothermal Orr-Sommerfeld equations (w and θ are perturbations of the vertical velocity and the temperature)

$$\begin{aligned}-i\omega\left(w''-\alpha^2 w\right)=i\alpha\left[\alpha^2U(y)w-U(y)w''+wU''(y)\right]-\\ -\frac{1}{Re}\left[w^{(4)}-2\alpha^2w''+\alpha^4w\right]-Ri\alpha^2\theta\end{aligned} \tag{13}$$

$$-i\omega\theta=-\left[i\alpha U(y)\theta+T'(y)w\right]+\frac{1}{Pe}\left[\theta''-\alpha^2\theta\right] \tag{14}$$

with the above made assumptions for α and ω. The profile $U(y)$ must be replaced by one of the profiles (11) depending on the problem considered.

We illustrate our approach on the case of inviscid and isothermal fluid. Then eq. (13) reduces to the Rayleigh equation

$$\left[U(y)-\frac{\omega}{\alpha}\right]\left(w''-\alpha^2w\right)-U''(y)w=0 \tag{15}$$

which can be considered as a linear eigenproblem for ω or a non-linear eigenproblem for -linear eigenproblem for α. For the temporal case $U = U_t(y)$, $\alpha = 1$, and (15) becomes

$$\left[f\left(y/\delta_v\right)-\omega\right]\left(w''-w\right)-f''\left(y/\delta_v\right)w=0 \qquad (16)$$

For the spatial case we have to account for eq.(12) and profile $U_s(y)$. Assuming additionally that for the spatial case $\alpha = 1+\beta$, where β is a complex correction to the unity wavenumber of the temporal case, we obtain

$$\left[\frac{1}{\lambda}\frac{\beta}{1+\beta}+f\left(\frac{y}{\delta_v}\right)\right]\left(w''-(1+\beta)^2 w\right)-f''\left(\frac{y}{\delta_v}\right)w=0 \qquad (17)$$

Assume monotonic temporal instability, i.e., $\omega = 0$ in eq.(16). Note that the Kelvin-Helmholz instability is monotonic, so this case is reprsentative. Then the eigenvector $w(y)$ of (16) coincides with the eigenvector of (17) for $\beta = 0$. The corresponding spatial instability can be calculated using the following iterative procedure:

$$\left[\frac{1}{\lambda}\frac{\beta_{k+1}}{1+\beta_k}+f\left(\frac{y}{\delta_v}\right)\right]\left(w''_{k+1}-\left(1+\beta_k\right)^2 w_{k+1}\right)-f''\left(\frac{y}{\delta_v}\right)w_{k+1}=0 \qquad (18)$$

where $\beta_0 = 0$ and each next β_{k+1} is chosen as the leading eigenvalue of the (k+1)-th eigenvalue problem (18). The perturbation (the eigenvector) corresponding to the temporal instability is calculated at the first iteration. The converged iterative process yields the spatial increment $\alpha = 1+\beta$ and the eigenvector of the spatial stability problem.

In the case of oscillatory temporal instability we calculate the time increment ω form the temporal problem (16) and then choose

$$\beta_0/\left[\lambda\left(1+\beta_0\right)\right]=-\omega \quad \Rightarrow \quad \beta_0=-\lambda\omega/\left(1+\lambda\omega\right) \qquad (19)$$

and then continue iterations (18).

It can be easily shown that for the general case, accounting for the viscosity and the stratification, the similar iterative process can be built. The resulting equations read

$$\begin{aligned}\left[\frac{1}{\lambda}\frac{\beta_{k+1}}{1+\beta_k}+f\left(\frac{y}{\delta_v}\right)\right]\left(w''_{k+1}-\left(1+\beta_k\right)^2 w_{k+1}\right)&=f''\left(\frac{y}{\delta_v}\right)w_{k+1}- \\ -\frac{2i}{Re\left(1+\beta_k\right)}\left[w^{(4)}_{k+1}-2\left(1+\beta_k\right)^2 w''_{k+1}+\left(1+\beta_k\right)^4 w_{k+1}\right]&+2iRi\left(1+\beta_k\right)\theta_{k+1}\end{aligned} \qquad (20)$$

$$\left[\frac{1}{\lambda}\frac{\beta_{k+1}}{1+\beta_k}+f\left(\frac{y}{\delta_v}\right)\right]\theta_{k+1}=\frac{2i}{1+\beta_k}T'(y)w-\frac{2i}{Pe\left(1+\beta_k\right)}\left[\theta''_{k+1}+\left(1+\beta_k\right)^2\theta_{k+1}\right] \qquad (21)$$

Numerical example

For an example we consider the case of inviscid isothermal fluid. The initial states are defined by $f(y/\delta_v) = \tanh(y/\delta_v)$ with $\delta_v = 0.5$ and $\lambda = 0.5$. The calculations were made by the global Galerkin method [2]. The convergence of parameter β is shown in the Table 1. The eigenvectors for the spatial and the temporal instability are compared in Fig.1.

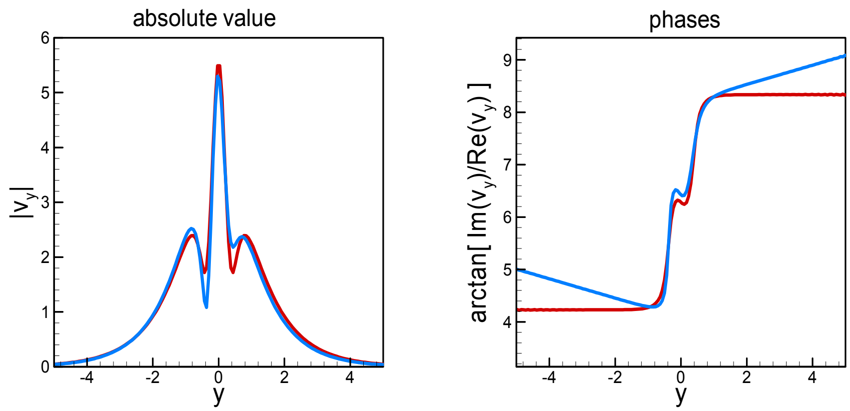

Fig.1. profiles of the perturbation of the vertical velocity. Red line – temporal problem, blue line – spatial problem. Calculation with 200 basis functions.

Concluding remarks

We have developed a simple iterative procedure yielding the solution of the spatial stability problem for the forced stratified mixing layer. The corresponding temporal case can be obtained as the first iteration of this procedure. Our preliminary calculations show that both spatial and temporal cases yield similar perturbation patterns, which allow us to compare results of full non-linear calculations made for both cases.

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References

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