Subcritical and supercritical bifurcations of the Benney equation

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Summary

In this paper we carry out the bifurcation analysis of the Benney equation (BE). The main result of this analysis shows that the primary bifurcation of the BE is supercritical, when the Reynolds number is below a certain critical value, and is subcritical, when the Reynolds number exceeds the latter. The subcritical structure of BE is verified numerically and further investigated analytically via a two-mode dynamical system. This truncation accurately describes solutions near the linear stability threshold and yields the exact transition to subcriticality. Furthermore, the analysis enables determination of a closed subdomain within the linearly stable region describing coexisting traveling waves (TW) predicted by the subcritical Hopf bifurcation. The waves of larger/smaller amplitude are found to be stable/unstable, respectively, and their transition to be defined by a saddle-node bifurcation.

Introduction

Benney [1] derived the nonlinear partial differential evolution equation referred to nowadays as the Benney equation. This evolution equation describes the nonlinear dynamics of an interface of a two-dimensional liquid film flowing on a fixed inclined plane. The BE has been extensively studied numerically and analytically over several decades. Lin [4] carried out a bifurcation analysis of the BE and found that the primary bifurcation is always supercritical. He also found that the filtered wave satisfying the pertinent complex Ginzburg-Landau equation (CGLE) is sideband-stable. However, our results presented below disagree with his results almost in all aspects.

Preliminaries

We study the Benney equation in the form given by [2]

$$h_t + 2h^2 h_x + \alpha[\frac{8R}{15}h^6 h_x + \frac{2S}{3}h^3 h_{xxx}]_x + O(\alpha^2) = 0, \tag{1}$$

which describes the spatiotemporal dynamics of the two-dimensional liquid film of a mean thickness d falling on a static vertical plate, when the physical properties of the liquid, such as density ρ, kinematic viscosity ν, and surface tension σ are specified. Here $h = h(x,t)$ represents the nondimensional film thickness depending on the dimensionless independent spatial and temporal variables x and t, respectively.

The system parameters include the fundamental gravity- and surface tension-related dimensionless parameters $R = gd^3/(2\nu^2)$, $W = \sigma/(\rho g d^2)$, which are Reynolds and inverse capillary numbers, respectively, and the small aspect-ratio parameter $\alpha = 2\pi d/\lambda$ being the

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ratio between the average film thickness d multiplied by 2π and the characteristic wavelength of the interfacial disturbances λ. The Reynolds and the rescaled inverse capillary numbers, R and $S = \alpha^2 W$, respectively, are assumed to be $O(1)$. The space-time variables (x,t) represent the corresponding physical space-time variables stretched by a small parameter α defined above. The wavelength λ is chosen as the entire length of the system, so that solution domain for Eq.1 is $0 \le x \le 2\pi$. In what follows the flow is considered in an infinite spatial range $(-\infty,\infty)$ with a 2π-periodicity.

The linearized version of Eq.1 around its trivial solution $h_0 \equiv 1$ reads in terms of a small disturbance of the flat film interface $u = h - 1$

$$u_t + 2u_x + \alpha(\frac{8R}{15}u_{xx} + \frac{2S}{3}u_{xxxx}) + O(\alpha^2) = 0. \tag{2}$$

To order $O(\alpha^2)$ Eq.2 has a TW solution with the fundamental wavenumber $k_0 = 1$ in the form $u(x,t) = \Gamma \exp[i(x-ct)] + \bar{\Gamma}\exp[-i(x-\bar{c}t)]$, where Γ is a complex amplitude of the wave independent of x,t, c is the complex wave celerity given by $c = c_r + ic_i,\ c_r = 2,\ c_i = \alpha(8R/15 - 2S/3)$, and bars denote complex conjugates. The solution $h = 1$ of the BE in the periodic domain $(0,2\pi)$ is asymptotically stable (unstable) if $c_i < 0$ ($c_i > 0$), which is equivalent to $8R/15 < 2S/3$ ($8R/15 > 2S/3$). The onset of instability is oscillatory via a Hopf bifurcation. At the instability threshold of the system $8R/15 = 2S/3$ the parameters of the problem are linked by the relationship $\alpha = \alpha_H \equiv \sqrt{4R/(5W)}$. Beyond the threshold, the film surface evolves as a stationary wave propagating downstream with the speed c_r.

Weakly nonlinear stability analysis of the Benney equation

We derive the Complex Ginzburg-Landau equation arising from the Benney equation 1. In the vicinity of criticality given by $c_i = O(\delta^2)$ which is equivalent to $8R/15 - 2S/3 = O(\delta^2)$ the critical wavenumber is $k = 1$, the slow independent variables are defined by $X = \delta x,\ T_1 = \delta t,\ T_2 = \delta^2 t$, where δ measures the distance from criticality, and the solution of Eq.1 is expanded in power series of δ as $h(\alpha,x,t,X,T_1,T_2) \equiv 1 + \delta\eta(\alpha,x,t,X,T_1,T_2) = 1 + \delta\eta_1 + \delta^2\eta_2 + \delta^3\eta_3 + \cdots$ with $\eta_j = \eta_j(\alpha,x,t,X,T_1,T_2),\ j = 1,2,3\cdots$.

Substituting these into Eq.1 and carrying out the solution order by order we obtain at $O(\delta)$

$$\eta_1 = \Gamma \exp[i(x - c_r t)] + \bar{\Gamma}\exp[-i(x - c_r t)], \tag{3}$$

where the amplitude $\Gamma = \Gamma(X,T_1,T_2)$ is to be determined and c_r is given above. At second order in δ we obtain that Γ has the functional form of $\Gamma = \Gamma(X - H_{2r}T_1, T_2)$, provided that $H_{2i} = O(\delta)$, where $H_2 \equiv H_{2r} + H_{2i} = 2 + 8i\alpha(2R/15 - S/3)$. Substituting Eq.3 into the equation obtained at third order in δ and eliminating its secular solution we obtain using MATHEMATICA the following equation related to the CGLE for the perturbation amplitude Γ:

$$\frac{\partial\Gamma}{\partial T_2}+i\nu\frac{\partial\Gamma}{\partial X}-c_i'\Gamma+(J_{1r}+iJ_{1i})\frac{\partial^2\Gamma}{\partial X^2}+(J_2+iJ_4)|\Gamma|^2\Gamma=0, \tag{4}$$

where $J_{1r}=-8\alpha R/3$, $J_2=5/2\alpha R-12\alpha R/5$, $J_4=-1$.

The character of the perturbation dynamics beyond the linear regime depends solely on the sign of J_2. When $J_2>0$, the saturation of the amplitude Γ is ensured. This is the case of a supercritical (forward) bifurcation. When $J_2<0$, the saturation does not occur (if higher order terms in Γ are not accounted for), and the corresponding case is that of a subcritical (inverted) bifurcation.

In view of the fact that the Benney equation to $O(\alpha)$ is being considered, one may be puzzled by the result obtained for J_2, as given above, due to the emergence of powers of α differing by two. However, the same result containing two terms of different signs is obtained when BE is transformed into the equation devoid of parameter α. Therefore, the bifurcation as predicted by the Benney equation is supercritical [subcritical] ($J_2>0$ [$J_2<0$]) if

$$R<R_c=\frac{5}{2\sqrt{6}\alpha}\ [R>R_c=\frac{5}{2\sqrt{6}\alpha}], \tag{5}$$

respectively. For instance, as follows from Eq.5 in the case of water films the domain of supercritical bifurcation is located along the Hopf curve for $0<R<R_c\approx 8.3894$, and is subcritical for $R>R_c$. This result will be verified below both theoretically and numerically.

It should be noted here that the critical value of R_c from Eqs.5 is $O(\alpha^{-1})$, This is formally outside the asymptotic range of $R=O(1)$ for which the BE was systematically derived from the Navier-Stokes equations. However, it represents a characteristic of the BE and is further verified to be within the bounds of the domain where solutions of the BE are bounded for a water film.

Numerical investigation of the Benney equation

In order to carry out a validation of our results obtained above for the nonlinear dynamics of water films, as described by the Benney equation, we use the relationship between d and α at the stability threshold obtained from Eq.5. These yield that for $d<d_c$ the primary bifurcation is supercritical, while for $d>d_c$ it is subcritical, where

$$d_c=(\frac{5\sqrt{5}}{2\sqrt{3}}\cdot\frac{\nu^3\sigma^{1/2}}{g^2\rho^{1/2}})^{2/11}. \tag{6}$$

Upon introduction of the physical properties of water into Eq.6 we find that $d_c=1.28976\times 10^{-2}$ cm.

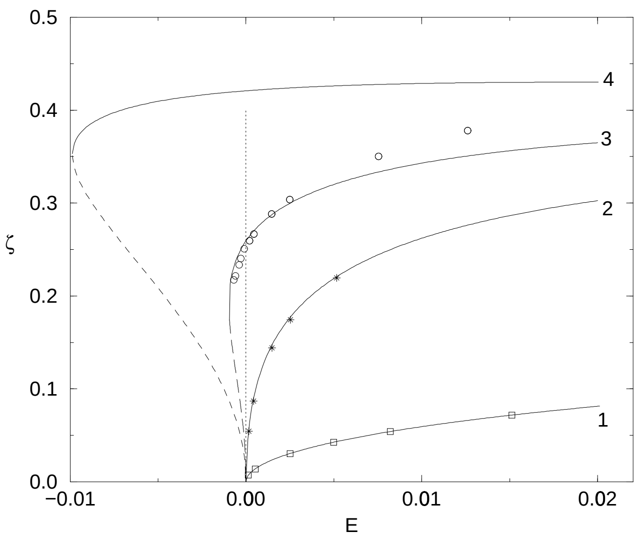

Figure 1: Variation of the wave amplitude ζ with E for various values of the thickness of the water film, as computed from Eq.1 and the modal Eqs.7, as $\zeta = 2\sqrt{a_1^2 + a_2^2}$. From Eq.1: squares- $d = 1.1 \times 10^{-2}$cm, $*$- $d = 1.28 \times 10^{-2}$cm, $\circ$- $d = 1.3 \times 10^{-2}$cm. The solid (dashed) curves computed from Eqs.7 correspond to stable (unstable) solutions: 1- $d = 1.1 \times 10^{-2}$cm, 2- $d = 1.28 \times 10^{-2}$cm, 3- $d = 1.3 \times 10^{-2}$cm, 4- $d = 1.32 \times 10^{-2}$cm. The vertical dotted line corresponds to the linear stability threshold, $E = 0$. The curves extending into the domain of $E < 0$ show subcritical bifurcation in support of our theoretical predictions.

Equation 1 is numerically solved along with periodic boundary conditions in the domain $0 \leq x \leq 2\pi$ and the parameters R, S calculated for water films of varying thicknesses d ranging around the critical value of d_c given by Eq.6. The results of such computations were recently presented in [5]. Both regimes of traveling stationary and nonstationary waves were found. Here we are concerned with numerical validation of our asymptotic analysis in the case of the primary bifurcation of BE.

Figure 1 displays the variation of the wave amplitude represented by the normalized peak-to-peak size of the wave $\zeta = (h_{max} - h_{min})/(h_{max} + h_{min})$ with the aspect ratio α, as presented in terms of the measure of the distance from criticality $E = (\alpha_H - \alpha)/\alpha_H$, where α_H is the critical value at the Hopf stability threshold for the specified film thickness d. Consistently with our theoretical predictions made above, the wave amplitude expressed by ζ tends to a nonzero value of $\zeta \approx 0.259$ when $\alpha \to \alpha_H$ in the case of $d = 1.3 \times 10^{-2}$cm. The branch of this curve that extends into the domain of negative E corresponding to $\alpha > \alpha_H$, which constitutes the linearly stable domain. This fact provides the numerical evidence of the subcritical solution and verification of the theoretical results, Eqs.5. Furthermore, the results corresponding to $d = 1.1 \times 10^{-2}$cm are consistent with the predicted supercritical character of the primary bifurcation and the value of ζ decreases to zero with decreasing E ($\zeta \to 0$ as $\alpha \to \alpha_H$). The curve corresponding to $d = 1.28 \times 10^{-2}$cm represents a supercritical case according to Eq.5 and bends down in the vicinity of $E = 0$, as if it were pointing to the reference point of the graph. However, we note that one cannot numerically resolve the question whether it actually hits the reference point. Thus, in order to further investigate the subcritical domain and its characteristics, we investigate a two-mode dynamical system. We note that the subcritical domain (located above the Hopf curve for $R > R_c$) is particularly difficult to determine as numerical continuation methods employed on pde's like Eq.1 are computer intensive.

Stability analysis of a truncated bimodal dynamical system

In a recent paper [3] demonstrated the validity of a low-order modal expansion with respect to the numerically solved BE. Furthermore, a two-mode model was found to coincide with the numerical solution along the Hopf curve separating the regions of linear stability and of bounded solutions of the BE. Thus, in order to further investigate the subcritical domain, we investigate a truncated two-mode dynamical system deduced from the BE.

Consider a solution of Eq.1 in the form of a truncated Fourier series $h(x,t) = 1 + \sum_{n=1}^{N} [z_n(t)\exp(inx) + \overline{z}_n(t)\exp(-inx)]$. Substituting this into Eq.1, truncating it to $N = 2$, representing the complex amplitudes in the form $z_n = a_n \exp(i\theta_n)$, and employing the phase relationship $\phi = 2\theta_1 - \theta_2$ yields to third order of nonlinearity

$$\begin{aligned} \dot{a}_1 &= \beta_{111}a_1 + (\beta_{121}\cos\phi - 4\sin\phi)a_1a_2 + a_1(\beta_{131}a_1^2 + \beta_{132}a_2^2), \\ \dot{a}_2 &= \beta_{211}a_2 + (\beta_{221}\cos\phi + 4\sin\phi)a_1^2 + a_2(\beta_{231}a_1^2 + \beta_{232}a_2^2), \\ \dot{\phi} &= (-2\beta_{121}\sin\phi - 8\cos\phi)a_2 + (-\beta_{221}\sin\phi + 4\cos\phi)\frac{a_1^2}{a_2} - 4(a_2^2 - a_1^2), \end{aligned} \tag{7}$$

where $\beta_{nji} = \alpha(M_{nji}R - N_{nji}S)$ and M_{nji}, N_{nji} are constants deduced from Eq.1.

Traveling waves of the system correspond to fixed points of the reduced polar Eqs.7 with a constant, nonzero phase difference ($\dot{a}_1 = \dot{a}_2 = \dot{\phi} = 0$). Close to the Hopf bifurcation we assume that the modal amplitudes are both small and ordered as $a_1 \to \varepsilon a_1,\ a_2 \to \varepsilon^2 a_2$. Thus, the phase evolution is dominated by the second term in the right-hand side of Eq.7c

$$\tan\phi^* = \frac{4}{\beta_{221}} = const + O(\varepsilon^2),\ \beta_{131}a_1^2 = -(\beta_{111} + \kappa_1 a_2) \tag{8}$$

where a_2 is determined from a quadratic equation

$$\kappa_1\beta_{231}a_2^2 + (\beta_{111}\beta_{231} - \beta_{131}\beta_{211} + \kappa_1\kappa_2)a_2 + \beta_{111}\kappa_2 = 0, \tag{9}$$

with $\kappa_1 = \beta_{121}\cos\phi^* - 4\sin\phi^*, \kappa_2 = \beta_{221}\cos\phi^* + 4\sin\phi^*$.

Recall that the Hopf bifurcation at the stability threshold is defined by $8R/15 = 2S/3$ which is $\beta_{111} = 0$. Consequently, the amplitude of the nonzero TW is deduced from Eq.9 to yield

$$a_2 = (\beta_{131}\beta_{211} - \kappa_1\kappa_2)\beta_{231}^{-1}\kappa_1^{-1}. \qquad (10)$$

A zero value of a_2 yields the boundary between the subcritical and supercritical bifurcations which is found to be identical to the criterion obtained in Eq.5.

The domain of existence of the subcritical TW is delineated by the zero value of the discriminant of Eq.9. Stability of the subcritical TW is determined from the Jacobian of Eqs.7 at the corresponding fixed points. The lower of the two TW yields a positive real eigenvalue corresponding to unstable saddle-foci, whereas the upper TW have negative real parts and are stable sinks. Therefore, as anticipated, the bifurcation points defined by the zero discriminant are saddle- nodes. We note that [6] found a similar subcritical domain of TW in their investigation of traveling waves of BE using AUTO software for water films. However, while our numerically obtained curve of unbounded solutions of the BE tends to the linear stability threshold curve from below, their curve for blow-up intersects the Hopf curve at $R \approx 8.9$.

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