Boundary Integral Equations with the Divergence Free Property for Elastostatics Problems

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Summary

The paper presents boundary integral equations (BIEs) with the divergence free property (DFP) for linear elastostatics problems. The application of the BIEs allows reducing two-dimensional (2D) problems of elastostatics to an evaluation of single-valued potential functions. Hence now it is possible for elastostatics problems to solve the general 2D-boundary element formulation analytically, and moreover to construct various kinds of 2D finite elements only based on the analytical solution omitting the numerical integration technique.

Introduction

The direct BE formulation for 2D elastostatic problems [1] can be derived from Bettis's reciprocal work theorem for two self-equilibrated states of displacements $\mathbf{u} \,|\, \mathbf{u}^*$, tractions $\mathbf{t} \,|\, \mathbf{t}^*$ and volume forces $\mathbf{b} \,|\, \mathbf{b}^*$. If Hooke's body is exposed to two different systems of volume and surface forces, then the actual work done by the forces of the first system along the displacements of the second system is equal to that work done by the forces of the second system along the displacements belonging to the first system:

$$\int_\Omega b_i^* \, u_i \, d\Omega + \int_\Gamma t_i^* \, u_i \, d\Gamma = \int_\Omega b_i \, u_i^* \, d\Omega + \int_\Gamma t_i \, u_i^* \, d\Gamma . \tag{1}$$

In Eq. (1) the displacements, tractions (i.e. the stress vectors $t_i = \sigma_{ij} n_j$ related to the outward normal vector, n_j) and body forces are respectively determined on the boundary $\Gamma = \partial\Omega$ and in the domain Ω. The field of $\mathbf{u}^*$, $\mathbf{t}^*$ and $\mathbf{b}^*$ corresponding to the governing solution of elasticity theory can be expressed as

$$u_i^*(p) = U_{ij}(p,q)\, \mathrm{e}_j(q) \tag{2a}$$

$$t_i^*(p) = T_{ij}(p,q)\, \mathrm{e}_j(q) \tag{2b}$$

$$b_i^*(p) = \delta(r)\, \mathrm{e}_i(q) \tag{2c}$$

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for the i direction at any field point p due to the unit force e_j in the j direction applied at the load point q. In Eqs (2) δ_{ij} is the Kronecker delta function, $\delta(r)$ the Dirac delta function, and the fundamental solutions for 2D linear elastic problems are given by

$$U_{ij}(p,q) = \frac{A_C}{2\mu}\left[(3-4\nu)\ln(r)\delta_{ij} - r_{,i}r_{,j}\right] \tag{3a}$$

$$T_{ij}(p,q) = \frac{A_C}{r}\left\{\left[(1-2\nu)\delta_{ij} + 2r_{,i}r_{,j}\right]\frac{\partial r}{\partial n} - (1-2\nu)(r_{,i}n_j - r_{,j}n_i)\right\} \tag{3b}$$

where $A_C = -1/\left[4\pi(1-\nu)\right]$, μ is the shear modulus of elasticity, ν Poisson's ratio and r is the distance between p and q.

Formulation of the BIEs with the DFP for Elasticity Problems

In the absence of volume forces Eq. (1) can be written as:

$$Q \in \Gamma \;\rightarrow\; u_i(p) = \int_\Gamma \left[U_{ij}(p,Q)\sigma_{jk}(Q) - T^*_{ijk}(p,Q)u_j(Q)\right]n_k(Q)d\Gamma\,. \tag{4}$$

If the DFP [2,3] of the boundary integrand, treated here as a vector field $\mathbf{F}$, is fulfilled:

$$(\nabla\cdot\mathbf{F})e_i = \left(U_{ij}\,\sigma_{jk} - T^*_{ijk}\,u_j\right)_{,k}e_i = \left[\left(U_{ij,k}\,\sigma_{jk} - T^*_{ijk}\,u_{j,k}\right) + \left(U_{ij}\,\sigma_{jk,k} - T^*_{ijk,k}\,u_j\right)\right]e_i = 0\,, \tag{5}$$

then the boundary integral is independent of the path chosen for the integration. Note a straightforward application of the equilibrium conditions leads to the following expressions: $\sigma_{jk,k} = 0$ and $\sigma^*_{jk,k} = T^*_{ijk,k}\,e_i = 0$, where e_i $(i = 1,2)$ are the unit vectors. However the use of Bettis's reciprocal work theorem yields

$$\left(U_{ij,k}\,\sigma_{jk} - T^*_{ijk}\,u_{j,k}\right)e_i = u^*_{j,k}\,\sigma_{jk} - \sigma^*_{jk}\,u_{j,k} = \varepsilon^*_{jk}\,\sigma_{jk} - \sigma^*_{jk}\,\varepsilon_{jk} = \varepsilon^*_{jk}\,k_{jkmn}\,\varepsilon_{mn} - \varepsilon_{jk}\,k_{jkmn}\,\varepsilon^*_{mn} = 0$$

where $\mathbf{k}$ represents the elasticity-tensor for a homogeneous isotropic medium.

On each boundary element $\Delta\Gamma_{(\beta)}$ (see Fig. 1), the displacement components $\tilde{u}_{\xi_1(\beta)}$ and $\tilde{u}_{\xi_2(\beta)}$ are approximated by the linear interpolation function. Note values of traction components $\tilde{t}_{\xi_1(\beta)}$ and $\tilde{t}_{\xi_2(\beta)}$ are constant along $\Delta\Gamma_{(\beta)}$. The formulated functions, which are needed to evaluate the boundary integrals, are enclosed in the appendix. A notation of used symbols is shown in Fig. 2.

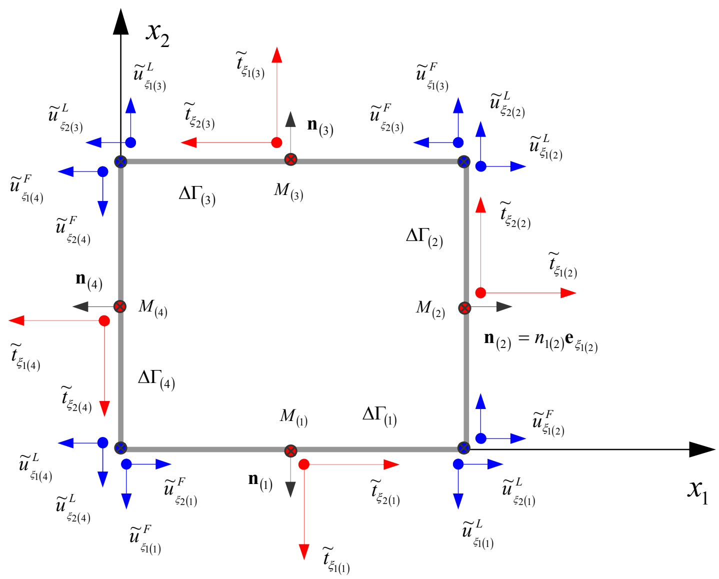

Fig. 1: Basic notations.

Conclusion

The presented here solution allows to more efficiently formulate so-called fast boundary (or finite) elements in solid mechanics.

Reference

1 Aliabadi, M. H. and Rooke D. P. (1992): *Numerical Fracture Mechanics*, Kluwer Academic Publishers.

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3 Nagarajan, A., Lutz E. and Mukherjee S. (1994): "A Novel Boundary Element Method for Linear Elasticity with no Numerical Integration for Two-Dimensional and Line Integrals for Three-Dimensional Problems", *ASME J. Appl. Mech.*, Vol. 61, pp. 264-269.

Appendix

Singular expressions for: $\xi_1 = 0$ and $\xi_2^F = -\Delta l/2$

$$\Delta\widetilde{G}_{\eta_1}(\Delta l) = \int_{-\Delta l/2}^{\Delta l/2} (3-4\nu)\ln(\xi_2)\widetilde{t}_{\xi_1}\, d\xi_2 =$$
$$(3-4\nu)\lim_{\varepsilon\to 0} 2\left[\xi_2(\ln(\xi_2)-1)\right]\Big|_{\varepsilon}^{\Delta l/2}\ \widetilde{t}_{\xi_1} = (3-4\nu)\left[\ln\left(\frac{\Delta l}{2}\right)-1\right]\Delta l\ \widetilde{t}_{\xi_1} \qquad \text{(A1)}$$

$$\Delta\widetilde{G}_{\eta_2}(\Delta l) = \int_{-\Delta l/2}^{\Delta l/2} \left[(3-4\nu)\ln(\xi_2)-1\right]\widetilde{t}_{\xi_2}\, d\xi_2 = \left\{(3-4\nu)\left[\ln\left(\frac{\Delta l}{2}\right)-1\right]-1\right\}\Delta l\ \widetilde{t}_{\xi_2} \qquad \text{(A2)}$$

$$\Delta\widetilde{H}_{\eta_1}^{F}(\Delta l) = \int_{-1}^{1} \frac{(1-2\nu)}{\xi_2^*}\widetilde{u}_{\xi_2}^{F}\frac{\Delta l}{2}\frac{(1-\zeta)}{2}d\zeta = \int_{-\Delta l/2}^{\Delta l/2} \frac{-(1-2\nu)}{\xi_2^*}\widetilde{u}_{\xi_2}^{F}\left(\frac{\xi_2^*}{\Delta l}-\frac{1}{2}\right)d\xi_2^* = -(1-2\nu)\widetilde{u}_{\xi_2}^{F} \qquad \text{(A3)}$$

$$\Delta\widetilde{H}_{\eta_1}^{L}(\Delta l) = \int_{-1}^{1} \frac{(1-2\nu)}{\xi_2^*}\widetilde{u}_{\xi_2}^{L}\frac{\Delta l}{2}\frac{(1+\zeta)}{2}d\zeta = \int_{-\Delta l/2}^{\Delta l/2} \frac{(1-2\nu)}{\xi_2^*}\widetilde{u}_{\xi_2}^{L}\left(\frac{\xi_2^*}{\Delta l}+\frac{1}{2}\right)d\xi_2^* = (1-2\nu)\widetilde{u}_{\xi_2}^{L} \qquad \text{(A4)}$$

$$\Delta\widetilde{H}_{\eta_2}^{F}(\Delta l) = \int_{-1}^{1} \frac{-(1-2\nu)}{\xi_2^*}\widetilde{u}_{\xi_1}^{F}\frac{\Delta l}{2}\frac{(1-\zeta)}{2}d\zeta = \int_{-\Delta l/2}^{\Delta l/2} \frac{(1-2\nu)}{\xi_2^*}\widetilde{u}_{\xi_1}^{F}\left(\frac{\xi_2^*}{\Delta l}-\frac{1}{2}\right)d\xi_2^* = (1-2\nu)\widetilde{u}_{\xi_1}^{F} \qquad \text{(A5)}$$

$$\Delta\widetilde{H}_{\eta_2}^{L}(\Delta l) = \int_{-1}^{1} \frac{-(1-2\nu)}{\xi_2^*}\widetilde{u}_{\xi_1}^{L}\frac{\Delta l}{2}\frac{(1+\zeta)}{2}d\zeta = \int_{-\Delta l/2}^{\Delta l/2} \frac{-(1-2\nu)}{\xi_2^*}\widetilde{u}_{\xi_1}^{L}\left(\frac{\xi_2^*}{\Delta l}+\frac{1}{2}\right)d\xi_2^* = -(1-2\nu)\widetilde{u}_{\xi_1}^{L} \qquad \text{(A6)}$$

because of: $\displaystyle\int_{-\Delta l/2}^{\Delta l/2}\frac{d\xi_2^*}{\xi_2^*} = \lim_{\varepsilon\to 0^+}\left(\int_{-\Delta l/2}^{-\varepsilon}\frac{d\xi_2^*}{\xi_2^*}+\int_{\varepsilon}^{\Delta l/2}\frac{d\xi_2^*}{\xi_2^*}\right) = \lim_{\varepsilon\to 0^+}\left[\ln(\varepsilon)-\ln(\Delta l/2)+\ln(\Delta l/2)-\ln(\varepsilon)\right] = 0$

Non-singular expressions for: $\xi_1 \neq 0$

$$\widetilde{G}_{\eta_1}(\xi_2) = \int\left\{\left[(3-4\nu)\ln\sqrt{\xi_1^{\,2}+\xi_2^{\,2}}-\frac{\xi_1^{\,2}}{\xi_1^{\,2}+\xi_2^{\,2}}\right]\widetilde{t}_{\xi_1}-\frac{\xi_1\xi_2}{\xi_1^{\,2}+\xi_2^{\,2}}\widetilde{t}_{\xi_2}\right\}d\xi_2 =$$
$$\left[(3-4\nu)\xi_2\left(\ln\sqrt{\xi_1^{\,2}+\xi_2^{\,2}}-1\right)+2(1-2\nu)\xi_1\arctan\frac{\xi_2}{\xi_1}\right]\widetilde{t}_{\xi_1}-\xi_1\ln\sqrt{\xi_1^{\,2}+\xi_2^{\,2}}\,\widetilde{t}_{\xi_2}+\widetilde{C}_{\eta_1} \qquad \text{(A7)}$$

$$\widetilde{G}_{\eta_2}(\xi_2) = \int\left\{-\frac{\xi_1\xi_2}{\xi_1^{\,2}+\xi_2^{\,2}}\widetilde{t}_{\xi_1}+\left[(3-4\nu)\ln\sqrt{\xi_1^{\,2}+\xi_2^{\,2}}-\frac{\xi_2^{\,2}}{\xi_1^{\,2}+\xi_2^{\,2}}\right]\widetilde{t}_{\xi_2}\right\}d\xi_2 =$$
$$-\xi_1\ln\sqrt{\xi_1^{\,2}+\xi_2^{\,2}}\,\widetilde{t}_{\xi_1}+\left[(3-4\nu)\xi_2\ln\sqrt{\xi_1^{\,2}+\xi_2^{\,2}}+4(1-\nu)\left(\xi_1\arctan\frac{\xi_2}{\xi_1}-\xi_2\right)\right]\widetilde{t}_{\xi_2}+\widetilde{C}_{\eta_2} \qquad \text{(A8)}$$

$$\begin{aligned}
&\widetilde{H}^F_{\eta_1}(\zeta)=\int\left\{\left[\frac{(1-2\nu)\xi_1}{\xi_1^{\,2}+\xi_2^{*2}}+\frac{2\xi_1^{\,3}}{\left(\xi_1^{\,2}+\xi_2^{*2}\right)^2}\right]\widetilde{u}^F_{\xi_1}+\left[\frac{(1-2\nu)\xi_2^*}{\xi_1^{\,2}+\xi_2^{*2}}+\frac{2\xi_1^{\,2}\xi_2^*}{\left(\xi_1^{\,2}+\xi_2^{*2}\right)^2}\right]\widetilde{u}^F_{\xi_2}\right\}\frac{\Delta l}{2}\frac{(1-\zeta)}{2}d\zeta=\\
&\left\{(1-2\nu)\left[\frac{-\xi_1A(\xi_2^*)}{\Delta l}+\left(\xi_2^F+\Delta l\right)B(\xi_2^*)\right]+\left(\xi_2^F+\Delta l\right)B(\xi_2^*)+C(\xi_2^*)\right\}\widetilde{u}^F_{\xi_1}+\\
&\left\{(1-2\nu)\left[\frac{\left(\xi_2^F+\Delta l\right)A(\xi_2^*)}{\Delta l}+\xi_1B(\xi_2^*)-\frac{\xi_2^F}{\Delta l}-\frac{(1+\zeta)}{2}\right]-\xi_1\left[B(\xi_2^*)+\frac{\xi_1}{\xi_1^{\,2}+\xi_2^{*2}}\frac{(1-\zeta)}{2}\right]\right\}\widetilde{u}^F_{\xi_2}+\widetilde{D}^F_{\eta_1}
\end{aligned}\qquad\text{(A9)}$$

$$\begin{aligned}
&\widetilde{H}^L_{\eta_1}(\zeta)=\int\left\{\left[\frac{(1-2\nu)\xi_1}{\xi_1^{\,2}+\xi_2^{*2}}+\frac{2\xi_1^{\,3}}{\left(\xi_1^{\,2}+\xi_2^{*2}\right)^2}\right]\widetilde{u}^L_{\xi_1}+\left[\frac{(1-2\nu)\xi_2^*}{\xi_1^{\,2}+\xi_2^{*2}}+\frac{2\xi_1^{\,2}\xi_2^*}{\left(\xi_1^{\,2}+\xi_2^{*2}\right)^2}\right]\widetilde{u}^L_{\xi_2}\right\}\frac{\Delta l}{2}\frac{(1+\zeta)}{2}d\zeta=\\
&\left\{(1-2\nu)\left[\frac{\xi_1A(\xi_2^*)}{\Delta l}-\xi_2^FB(\xi_2^*)\right]-\xi_2^FB(\xi_2^*)-\xi_1\frac{\xi_1^{\,2}+\xi_2^F\xi_2^*}{\left(\xi_1^{\,2}+\xi_2^{*2}\right)\Delta l}\right\}\widetilde{u}^L_{\xi_1}+\\
&\left\{(1-2\nu)\left[\frac{-\xi_2^FA(\xi_2^*)}{\Delta l}-\xi_1B(\xi_2^*)+\frac{\xi_2^F}{\Delta l}+\frac{(1+\zeta)}{2}\right]+\xi_1\left[B(\xi_2^*)-\frac{\xi_1}{\xi_1^{\,2}+\xi_2^{*2}}\frac{(1+\zeta)}{2}\right]\right\}\widetilde{u}^L_{\xi_2}+\widetilde{D}^L_{\eta_1}
\end{aligned}\qquad\text{(A10)}$$

$$\begin{aligned}
&\widetilde{H}^F_{\eta_2}(\zeta)=\int\left\{\left[\frac{-(1-2\nu)\xi_2^*}{\xi_1^{\,2}+\xi_2^{*2}}+\frac{2\xi_1^{\,2}\xi_2^*}{\left(\xi_1^{\,2}+\xi_2^{*2}\right)^2}\right]\widetilde{u}^F_{\xi_1}+\left[\frac{(1-2\nu)\xi_1}{\xi_1^{\,2}+\xi_2^{*2}}+\frac{2\xi_1\xi_2^{*2}}{\left(\xi_1^{\,2}+\xi_2^{*2}\right)^2}\right]\widetilde{u}^F_{\xi_2}\right\}\frac{\Delta l}{2}\frac{(1-\zeta)}{2}d\zeta=\\
&\left\{(1-2\nu)\left[\frac{-\left(\xi_2^F+\Delta l\right)A(\xi_2^*)}{\Delta l}-\xi_1B(\xi_2^*)+\frac{\xi_2^F}{\Delta l}+\frac{(1+\zeta)}{2}\right]-\xi_1\left[B(\xi_2^*)+\frac{\xi_1}{\xi_1^{\,2}+\xi_2^{*2}}\frac{(1-\zeta)}{2}\right]\right\}\widetilde{u}^F_{\xi_1}+\\
&\left\{(1-2\nu)\left[\frac{-\xi_1A(\xi_2^*)}{\Delta l}+\left(\xi_2^F+\Delta l\right)B(\xi_2^*)\right]-\frac{2\xi_1A(\xi_2^*)}{\Delta l}+\left(\xi_2^F+\Delta l\right)B(\xi_2^*)-C(\xi_2^*)\right\}\widetilde{u}^F_{\xi_2}+\widetilde{D}^F_{\eta_2}
\end{aligned}\qquad\text{(A11)}$$

$$\begin{aligned}
&\widetilde{H}^L_{\eta_2}(\zeta)=\int\left\{\left[\frac{-(1-2\nu)\xi_2^*}{\xi_1^{\,2}+\xi_2^{*2}}+\frac{2\xi_1^{\,2}\xi_2^*}{\left(\xi_1^{\,2}+\xi_2^{*2}\right)^2}\right]\widetilde{u}^L_{\xi_1}+\left[\frac{(1-2\nu)\xi_1}{\xi_1^{\,2}+\xi_2^{*2}}+\frac{2\xi_1\xi_2^{*2}}{\left(\xi_1^{\,2}+\xi_2^{*2}\right)^2}\right]\widetilde{u}^L_{\xi_2}\right\}\frac{\Delta l}{2}\frac{(1+\zeta)}{2}d\zeta=\\
&\left\{(1-2\nu)\left[\frac{\xi_2^FA(\xi_2^*)}{\Delta l}+\xi_1B(\xi_2^*)-\frac{\xi_2^F}{\Delta l}-\frac{(1+\zeta)}{2}\right]+\xi_1\left[B(\xi_2^*)-\frac{\xi_1}{\xi_1^{\,2}+\xi_2^{*2}}\frac{(1+\zeta)}{2}\right]\right\}\widetilde{u}^L_{\xi_1}+\\
&\left\{(1-2\nu)\left[\frac{\xi_1A(\xi_2^*)}{\Delta l}-\xi_2^FB(\xi_2^*)\right]+\frac{2\xi_1A(\xi_2^*)}{\Delta l}-\xi_2^FB(\xi_2^*)+\xi_1\frac{\xi_1^{\,2}+\xi_2^F\xi_2^*}{\left(\xi_1^{\,2}+\xi_2^{*2}\right)\Delta l}\right\}\widetilde{u}^L_{\xi_2}+\widetilde{D}^L_{\eta_2}
\end{aligned}\qquad\text{(A12)}$$

where $A(\xi_2^*) = \ln\sqrt{\xi_1^{\,2} + \xi_2^{*2}}$, $B(\xi_2^*) = \dfrac{1}{\Delta l}\arctan\dfrac{\xi_2^*}{\xi_1}$,

$C(\xi_2^*) = \xi_1 \dfrac{\xi_1^{\,2} + (\xi_2^F + \Delta l)\xi_2^*}{(\xi_1^{\,2} + \xi_2^{*2})\Delta l}$, $\xi_2^* = f_{\xi_2^*}(\zeta) = \xi_2^F + \dfrac{(1+\zeta)}{2}\Delta l$,

$\widetilde{C}_{\eta_1}$, $\widetilde{C}_{\eta_2}$, $\widetilde{D}_{\eta_1}^F$, $\widetilde{D}_{\eta_1}^L$, $\widetilde{D}_{\eta_2}^F$ and $\widetilde{D}_{\eta_2}^L$ are constants, and ($-1 \le \zeta \le 1$).

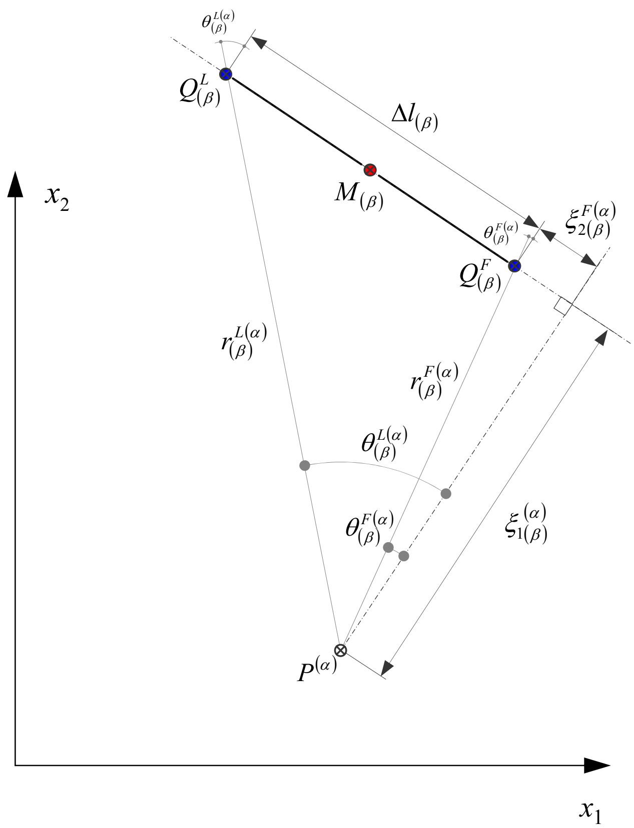

Fig. 2: Local co-ordinates system for the boundary element $\Delta\Gamma_{(\beta)}$ and source point $P^{(\alpha)}$.

Note that $\left|\theta_{(\beta)}^{L(\alpha)} - \theta_{(\beta)}^{F(\alpha)}\right| = \pi$ for the singular elements ($\xi_{1(\beta)}^{(\alpha)} = 0$) when $P^{(\alpha)} = M_{(\beta)}$.