BEM Anisotropic Thermoelastic Analysis of Bi-material Interface Cracks

C. L. Tan[1] and Y. C. Shiah[2]

Summary

In this paper, the thermo-elastostatic problem of an interface crack between two dissimilar anisotropic media is treated by the boundary element method (BEM). In a sequentially coupled manner, the direct domain mapping technique is applied to solve the thermal field in the presence of an interface crack. The associated thermoelastic stress field is then solved with the boundary integral equation which has the volume integral due to thermal effects transformed exactly into surface integrals. Special crack-tip elements which incorporate the oscillatory stress singularity are used for the analysis of the interface crack problem. Two numerical examples are presented to demonstrate the veracity of the formulation.

Introduction

Solutions using the BEM for coupled thermoelastic, bimateral interface crack problems in heterogeneous anisotropic media are extremely scarce indeed in the literature. This is in spite of the fact that this numerical technique has been well established as an efficient tool for linear elastic fracture mechanics, and special crack-tip elements which incorporate the proper oscillatory traction singularity had in fact been developed for treating interface cracks between anisotropic bodies [1]. A key reason for this is that thermoelastic effects manifest themselves as an additional volume integral term in the boundary integral equation (BIE) when using the direct formulation.

The volume integral due to thermal effects destroys the distinctive feature of the BEM as a truly boundary solution numerical technique for engineering analysis. A number of schemes have been proposed over the years, e.g. [2]-[5], to deal with this issue. The one that is perhaps most appealing fundamentally is the exact transformation method (ETM) where it is transformed analytically without approximations into boundary integrals, as has been developed in isotropy. For anisotropic thermoelasticity, however, this was not achieved until quite recently by the present authors [4]. A direct domain mapping technique was used to reduce the governing field equation for anisotropic steady-state heat conduction to Laplace's equation, in order to facilitate the volume-to-surface integral transformation. The approach was subsequently applied to crack problems in

[1] Dept. of Mechanical & Aerospace Engineering, Carleton University, Ottawa, Canada K1S 5B6
[2] Dept. of Aeronautical Engineering, Feng Chia University, Taichung, Taiwan, R.O.C.

homogeneous anisotropic bodies [5]. For heterogeneous domains with piecewise homogeneous sub-regions, distortion of these sub-regions in the mapped plane poses additional complications; relations invoking the conditions of equilibrium and compatibility of the steady state heat conduction at the interfaces between them had to be developed in the numerical formulation [6].

The main objective of this paper is to demonstrate the applicability of the general ETM scheme to treat interface crack problems between anisotropic media in the presence of a non-uniform temperature gradient. This has not been shown previously. To this end, two examples will be provided. First, however, the basic equations in the BIE formulation and the procedure to determine the fracture parameters of bi-material interface cracks in the BEM analysis are described.

Review of the BIE for anisotropic thermoelasticity

For the sake of brevity, only the key equations of the BIE for anisotropic thermoelasticity are presented here; the reader is referred to Ref. [4] for more details of the formulation. In the direct formulation of the BEM for an anisotropic solid in two-dimensions, the BIE which relates the displacements, u_i, and the tractions, t_i, on the boundary S of the domain Ω, may be written as

$$C_{ij}u_i(P) + \oint_S u_i(Q)T_{ij}(P,Q)dS = \oint_S t_i(Q)U_{ij}(P,Q)dS + \oint_S \gamma_{ik}n_k \Theta U_{ij}(P,Q)dS - \int_\Omega \gamma_{ik}\Theta_{,k}U_{ij}(P,q)d\Omega \tag{1}$$

where U_{ij}, and $T_{ij}(P,Q)$ are the displacement and traction fundamental solutions, respectively; Θ is the temperature change, and γ_{ik} are the coefficients related to the thermal properties of the body; and n_k is the unit outward normal at Q on S. The differential equation for steady state heat conduction is given by

$$K_{ij}\Theta_{ij} = -C_o \tag{2}$$

where K_{ij} are the thermal conductivity coefficients and C_o represent the uniform internal heat source. In the domain mapping technique, the method of characteristics is employed to convert this equation into Poisson's equation in a mapped Cartesian plane, thus

$$\Theta_{,\underline{ii}} = C_1 \tag{3}$$

where the underline in the indices denotes the mapped coordinate system. The right hand side of the equation becomes $C_1 = C_0 K_{11}/\Delta$, where $\Delta = (K_{11}K_{22} - K_{12}^2)$. Using Green's theorem and following the usual limiting process in BEM, the volume integral in Eq. (1) can be transformed into surface integrals in the mapped plane, and the resulting BIE for a simply-connected domain becomes

$$C_{ij}\,u_i(P) + \int_S u_i(Q)\,T_{ij}(P,Q)\;dS = \int_S t_i(Q)\,U_{ij}(P,Q)\;dS + \int_S \gamma_{ik}\,n_k\,\Theta\,U_{ij}(P,Q)\;dS + \int_{\hat{S}} [(\gamma_{\underline{ik}}\,Q_{\underline{ijk},\underline{t}}\,\Theta - \gamma_{\underline{ik}}\,Q_{\underline{ijk}}\,\Theta_{,\underline{t}} + C_l\gamma_{\underline{ik}}\,R_{\underline{ijkt}})\,n_{\underline{t}} - \gamma_{\underline{ik}}\,U_{\underline{ij}}(P,Q)\Theta\,n_k]\,d\hat{S} \quad (4)$$

where $\gamma_{\underline{ij}}$ now comprise of terms involving γ_{ij} and K_{ij}; and the functions $Q_{\underline{ijk}}$ and $R_{\underline{ijkt}}$ involve the material constants, the generalized complex variable z_i and the characteristic roots μ_i defined in the mapped plane.

Note that to solve Eq. (4), the temperature field in the anisotropic domain must first be obtained; this can be done now with a BEM code for simple potential theory, albeit first in the mapped plane as in Eq. (3). When treating a body consisting of two bonded media with dissimilar properties, the proper interface conditions of compatibility and equilibrium of the temperature field between the two adjacent materials (1) and (2) must further be invoked, as follows [6],

$$\Theta^{(1)} = \Theta^{(2)} \quad (5)$$

$$\Delta^{(1)}/\omega^{(1)}K_{11}^{(1)}\cdot(d\Theta^{(1)}/d\hat{n}^{(1)}) + \Delta^{(2)}/\omega^{(2)}K_{11}^{(2)}\cdot(d\Theta^{(2)}/d\hat{n}^{(2)}) = 0 \quad (6)$$

Equation (4) may be discretised and solved in the usual manner in BEM analysis. The main advantage of the approach here is that not only is it capable of dealing with general temperature distributions without incurring further numerical approximations as in some other schemes, it can also be directly applied to fracture problems. In the present work, the quadratic isoparametric element formulation is employed. For an interface crack between the dissimilar bodies, special quarter-point crack-tip elements which incorporate the appropriate oscillatory singularity are used and the coupled stress intensity factors, K_1 and K_2, may be accurately obtained using the "traction formulas" derived by Tan *et al* [1].

Numerical examples

Figure 1 presents the first example, Problem (a), considered. It is a plate made of two sub-regions with a central slant crack at the interface that has an orientation of 45° with respect to the global axes. The two sub-regions are made of a single crystal Al_2O_3 but with their principal axes oriented at angles, ±θ respectively, with respect to the global axes as shown. Following the usual notations but with asterisks denoting values in the directions of the principal axes, the mechanical properties of this crystal are listed below [7],

$\frac{E_{22}^*}{E_{11}^*}$	$\frac{E_{33}^*}{E_{11}^*}$	$\frac{G_{12}^*}{E_{11}^*}$	$\frac{G_{23}^*}{E_{11}^*}$	$\frac{G_{31}^*}{E_{11}^*}$	$\frac{\alpha_{22}^*}{\alpha_{11}^*}$	ν_{12}^*	ν_{13}^*	ν_{23}^*	$\eta_{12,1}^*$	$\eta_{12,2}^*$	$\eta_{12,3}^*$	$\frac{K_{22}^*}{K_{11}^*}$
$\frac{516}{345}$	$\frac{345}{345}$	$\frac{173}{345}$	$\frac{173}{345}$	$\frac{127}{345}$	$\frac{5.841}{5.171}$	.131	.362	.196	.59	0	-.59	$\frac{23.1}{25.2}$

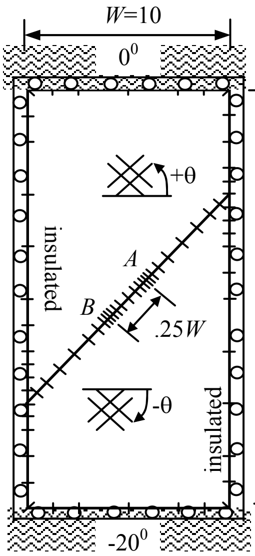

Fig. 1: An interface slant crack between dissimilar anisotropic media – Prob. (a)

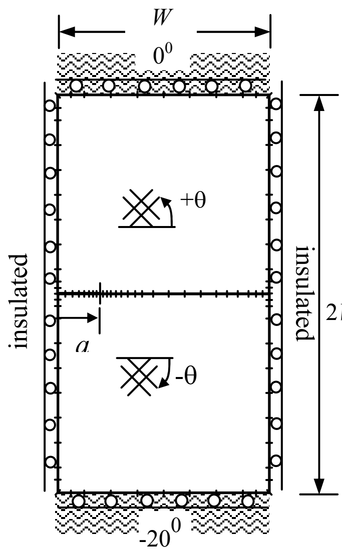

Fig. 2: An interface edge crack of a bonded material – Prob. (b)

The boundary conditions are as shown in the figure; plane stress conditions as well as no heat flux across the crack faces are assumed. Also shown is the BEM mesh discretisation employed. The well-established principle of superposition for stress intensity factors (SIF's) is used to check the validity of the solutions obtained. It involves two sub-problems of the same physical problem, one is uncracked and subjected to the original load conditions; the other is the cracked body, but subjected to only tractions at the crack faces corresponding to the stresses acting there in the first sub-problem. Table 1 lists results of the normalised stress intensity factors, K_1/K_o and K_2/K_o, where $K_o = E^*_{11}\alpha^*_{11}\Delta T\sqrt{(\pi a)}$, $\Delta T = 20^o$, for the two crack tips, *A* and *B*. The characteristic dimension used in the computation of the SIF's is the length of the interface crack. As can be seen, agreement between the SIF solutions obtained directly with the thermoelastic BEM analysis and those obtained by superposition is very good indeed. As a further verification of the algorithm for anisotropic analysis, the problem was also analysed as a quasi-isotropic case with almost identical properties in both Cartesian directions, and the results compared with those obtained through BEM isotropic analysis. The anisotropic quasi-isotropic results for K_1/K_o and K_2/K_o at tips *A* and *B* are 0.523 and 0.039, respectively; the corresponding values from a truly isotropic analysis are 0.522 and 0.028.

The second example, Problem (b), is shown in Fig. 2. It has a horizontal interface edge crack between the two bonded materials. The same single crystal alumina as in the previous example is considered and stress intensity factors for a range of crack sizes are obtained. A typical BEM mesh employed is also shown in Fig.2 and the characteristic dimension for the computation of the stress intensity

Table 1: Normalised stress intensity factors for Problem (a).
$(K_1/K_o)_S$, $(K_2/K_o)_S$- Superposition; $(K_1/K_o)_D$, $(K_2/K_o)_D$- Direct BEM

		θ=15⁰	θ=30⁰	θ=45⁰	θ=60⁰	θ=75⁰
Tip *A*	$(K_1/K_o)_S$	0.661	0.640	0.524	0.596	0.617
	$(K_1/K_o)_D$	0.678	0.623	0.532	0.586	0.615
	%Diff.	2.5	2.7	1.6	1.7	0.4
Tip *B*	$(K_1/K_o)_S$	0.630	0.600	0.507	0.556	0.586
	$(K_1/K_o)_D$	0.642	0.587	0.514	0.544	0.584
	% Diff.	1.9	2.1	1.3	2.0	0.3
Tip *A*	$(K_2/K_o)_S$	0.031	0.017	0.021	0.067	0.029
	$(K_2/K_o)_D$	0.031	0.015	0.026	0.072	0.033
	% Diff.	0.6	12.9	22.0	7.7	13.8
Tip *B*	$(K_2/K_o)_S$	0.126	0.112	0.006	0.127	0.166
	$(K_2/K_o)_D$	0.134	0.104	0.006	0.136	0.170
	% Diff.	6.7	7.0	3.6	7.1	2.4

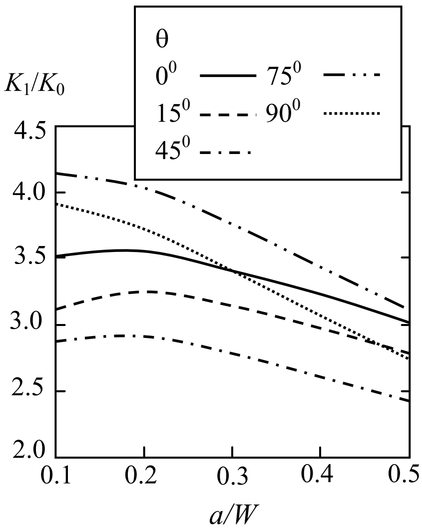

Fig. 3: Variation of K_1/K_0 with relative crack size – Prob. (b)

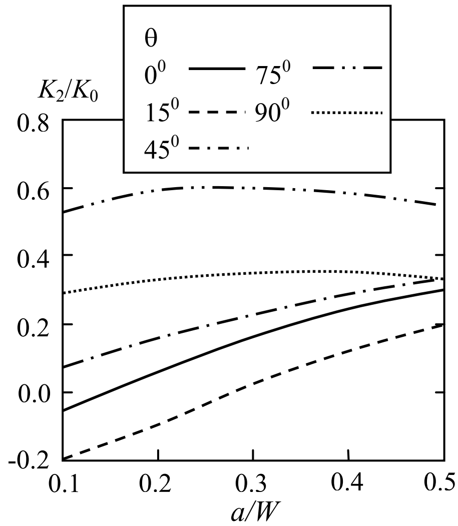

Fig. 4: Variation of K_2/K_0 with relative crack size – Prob. (b)

factors is again taken to be the crack length. The variations of the computed normalised SIF's with relative crack length are shown in Figs. 3 and 4 for a range of orientation angles of the material principal axes.

Conclusion

The determination of stress intensity factors using the BEM for an interface crack between dissimilar anisotropic media subjected to non-uniform temperature distribution has been presented in this paper. In the boundary integral equation formulation, the volume integral associated with thermal effects has been transformed exactly into surface integrals using a direct domain mapping technique. Its direct application to interface crack problems has been illustrated by two numerical examples.

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