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Normal Modes of Rotating Timoshenko Beams

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Summary

A modeling method for flapwise and chordwise bending vibration analysis for rotating Timoshenko beams is introduced. For the modeling method shear and the rotary inertia effects are correctly judged based on *Timoshenko beam theory*. Equations of motion of continuous models are derived from a modeling method which employs hybrid deformation variables. The equations thus derived are transmitted into dimensionless forms. The effects of dimensionless parameters on the modal characteristics of the Timoshenko beams are successfully examined through numerical study. In particular, eigenvalue loci veering phenomena and integrated mode shape critical deviations are contemplated and examined in this work.

Introduction

Rotating structures frequently occur in several types of engineering structures such as turbines, turbo machines and aircraft rotary wings. In order to design the rotating structures properly, there modal characteristics must be computed exactly. In the present study, the equations of motion of rotating Timoshenko beams are derived, using hybrid deformation variables introduced in [1], [2] and [3]. The critical use of hybrid deformation variables which distinguishes the present modeling method from other traditional modeling methods, is the key ingredient used to derive the equations of motion rigorously. Moreover, the combined effect of angular speed, hub radius, slenderness ratio, shear/extension modulus ratio on the shear and rotary inertia (thus, on the modal characteristics) of Timoshenko beams is successfully investigated in this study.

Equations of motion

In the present work, we use s, the arc length stretch, instead of u_1 to measure the displacement in the axial direction (see [1] for detail). We use u_{2b} and u_{2s} as deformations due to bending and shear respectively along the x_2 direction, and u_{3b} and u_{3s} as deformations due to bending and shear, respectively along the x_3 direction

The assumed mode approach will be used in the present study to investigate the natural frequencies and normal modes of Timoshenko beams. By employing the *Rayleigh Ritz method* the deformation variables are approximated as follows

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$$s(x,t)=\sum_{i=1}^{\mu_1}\phi_{1i}(x)q_{1i}(t)=\sum_{j=1}^{\mu_1}\phi_{1j}(x)q_{1j}(t) \tag{1}$$

$$u_{2b}(x,t)=\sum_{i=1}^{\mu_{2b}}\phi_{2bi}(x)q_{2bi}(t)=\sum_{j=1}^{\mu_{2b}}\phi_{2bj}(x)q_{2bj}(t) \tag{2}$$

$$u_{2s}(x,t)=\sum_{i=1}^{\mu_{2s}}\phi_{2si}(x)q_{2si}(t)=\sum_{j=1}^{\mu_{2s}}\phi_{2sj}(x)q_{2sj}(t) \tag{3}$$

$$u_{3b}(x,t)=\sum_{i=1}^{\mu_{3b}}\phi_{3bi}(x)q_{3bi}(t)=\sum_{j=1}^{\mu_{3s}}\phi_{3bj}(x)q_{3bj}(t) \tag{4}$$

$$u_{3s}(x,t)=\sum_{i=1}^{\mu_{3b}}\phi_{3si}(x)q_{3si}(t)=\sum_{j=1}^{\mu_{3s}}\phi_{3sj}(x)q_{3sj}(t) \tag{5}$$

where ϕ_{1i}, ϕ_{2bi}, ϕ_{2si}, ϕ_{3bi} and ϕ_{3si} are shape functions for $s, u_{2b}, u_{2s}, u_{3b}$ and u_{3s} respectively, and q_i's are generalized coordinates.

The equations of motion of a beam can be derived from the Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\left(\frac{\partial T}{\partial q_i}\right)+\frac{\partial U}{\partial q_i}=0 \quad i=1,2,\ldots,N \tag{6}$$

in which, $N=\mu_1+\mu_{2b}+\mu_{2s}+\mu_{3b}+\mu_{3s}$ is the total number of generalized coordinates, and U and T are the strain and kinetic energies of the beam, respectively,

$$\begin{aligned} U=\frac{1}{2}\int_0^L\Bigg[&EA\left(\frac{\partial s}{\partial x}\right)^2+EI_3\left(\frac{\partial^2 u_{2b}}{\partial x^2}\right)^2+EI_2\left(\frac{\partial^2 u_{3b}}{\partial x^2}\right)^2\\ &+\frac{\mu AG}{2}\left(\left(\frac{\partial u_{2s}}{\partial x}\right)^2+\left(\frac{\partial u_{3s}}{\partial x}\right)^2\right)\Bigg]dx \end{aligned} \tag{7}$$

$$T=\int_0^L\rho\vec{a}\cdot\frac{\partial\vec{v}}{\partial\dot{q}_i}dx+\frac{1}{2}\int_0^L\left[\frac{\rho I_3}{A}\left(\frac{\partial^2 u_{2b}}{\partial x\partial t}\right)^2+\frac{\rho I_2}{A}\left(\frac{\partial^2 u_{3b}}{\partial x\partial t}\right)^2\right]dx \tag{8}$$

where E is Young's modulus, A the cross-sectional area of the beam, μ the shear co-efficient, G the shear modulus, I_2 and I_3 the second area moments of inertia, respectively, L the length of the beam, $\vec{v}$ and $\vec{a}$ the the velocity and acceleration vectors, respectively. The velocity vector can be obtain as

$$\begin{aligned} \vec{v}=\{&\dot{s}-\int_0^x\left[\left(u'_{2b}+u'_{2s}\right)\left(\dot{u}'_{2b}+\dot{u}'_{2s}\right)+\left(u'_{3b}+u'_{3s}\right)\left(\dot{u}'_{3b}+\dot{u}'_{3s}\right)\right]d\sigma-\Omega u_{2b}-\Omega u_{2s}\}\vec{i}_1\\ &+\left[\dot{u}_{2b}+\dot{u}_{2s}+\Omega\left[r+x+\left(s-\frac{1}{2}\int_0^x\left[\left(u'_{2b}+u'_{2s}\right)^2+\left(u'_{3b}+u'_{3s}\right)^2\right]d\sigma\right)\right]\right]\vec{i}_2\\ &+\left(\dot{u}_{3b}+\dot{u}_{3s}\right)\vec{i}_3 \end{aligned} \tag{9}$$

As the present study will be focusing on beams with constant cross-section, it is useful to rewrite the equations of motion in a dimensionless form. For this transformation several dimensionless variables and parameters are defined as follows:

$$\tau = \frac{t}{T_0}, \qquad \xi = \frac{x}{L}, \qquad \theta_{ai} = \frac{q_{ai}}{L},$$
$$\gamma = \Omega T_0, \quad \delta = \frac{r}{L}, \qquad \kappa = \frac{I_2}{I_3} \tag{10}$$

where, T_0 is a time parameter, defined as

$$T_0 = \sqrt{\frac{\rho L^4}{EI_3}} \tag{11}$$

Substituting Eqs. (10) and (1) - (5) into the Lagrange's equation, and ignoring the coupling effect between bending and axial-stretching, we can obtain the following linearized equations of motion for bending and shear.

$$\sum_{j=1}^{\mu_{2b}} \left(M_{ij}^{2b2b} + \frac{1}{\beta} M_{ij}^{RI_3 2b2b} \right) \ddot{\theta}_{2bj} + \sum_{j=1}^{\mu_{2s}} M_{ij}^{2b2s} \ddot{\theta}_{2sj}$$
$$+ \sum_{j=1}^{\mu_{2b}} \left\{ \gamma^2 \left(K_{ij}^{GB2b2b} + \delta K_{ij}^{GA2b2b} - M_{ij}^{2b2b} \right) + K_{ij}^{B_3 2b2b} \right\} \theta_{2bj}$$
$$+ \sum_{j=1}^{\mu_{2s}} \gamma^2 \left(K_{ij}^{GB2b2s} + \delta K_{ij}^{GA2b2s} - M_{ij}^{2b2s} \right) \theta_{2sj} = 0 \tag{12}$$

$$\sum_{j=1}^{\mu_{2b}} M_{ij}^{2s2b} \ddot{\theta}_{2bj} + \sum_{j=1}^{\mu_{2s}} M_{ij}^{2s2s} dx \ddot{\theta}_{2sj} + \sum_{j=1}^{\mu_{2b}} \gamma^2 \left(K_{ij}^{GB2s2b} + \delta K_{ij}^{GA2s2b} - M_{ij}^{2s2b} \right) \theta_{2bj}$$
$$+ \sum_{j=1}^{\mu_{2s}} \left\{ \gamma^2 \left(K_{ij}^{GB2s2s} + \delta K_{ij}^{GA2s2s} - M_{ij}^{2s2s} \right) + \beta \eta K_{ij}^{S^* 2s2s} \right\} \theta_{2sj} = 0 \tag{13}$$

$$\sum_{j=1}^{\mu_{3b}} \left(M_{ij}^{3b3b} + \frac{1}{\beta} M_{ij}^{RI_2 3b3b} \right) \ddot{\theta}_{3bj} + \sum_{j=1}^{\mu_{3s}} M_{ij}^{3b3s} \ddot{\theta}_{3sj}$$
$$+ \sum_{j=1}^{\mu_{3b}} \left\{ \gamma^2 \left(K_{ij}^{GB3b3b} + \delta K_{ij}^{GA3b3b} \right) + K_{ij}^{B_2 3b3b} \right\} \theta_{3bj}$$
$$+ \sum_{j=1}^{\mu_{3s}} \gamma^2 \left(K_{ij}^{GB3b3s} + \delta K_{ij}^{GA3b3s} \right) \theta_{3sj} = 0 \tag{14}$$

$$\sum_{j=1}^{\mu_{3b}} M_{ij}^{3s3b}\ddot{\theta}_{3bj} + \sum_{j=1}^{\mu_{3s}} M_{ij}^{3s3s} dx \ddot{\theta}_{3sj} + \sum_{j=1}^{\mu_{3b}} \gamma^2 \left(K_{ij}^{GB3s3b} + \delta K_{ij}^{GA3s3b}\right)\theta_{3bj}$$
$$+ \sum_{j=1}^{\mu_{3s}} \left\{\gamma^2 \left(K_{ij}^{GB3s3s} + \delta K_{ij}^{GA3s3s}\right) + \beta\eta K_{ij}^{S^*3s3s}\right\}\theta_{3sj} = 0 \tag{15}$$

where, β and η are the square of the slenderness ratio and the material ratio, respectively, defined as

$$\beta = \frac{AL^2}{I_3}, \quad \eta = \frac{\mu G}{E} \tag{16}$$

and

$$M_{ij}^{mn} = \int_0^1 \psi_{mi}\psi_{nj} d\xi \tag{17}$$
$$M_{ij}^{RI_3mn} = \int_0^1 \psi'_{mi}\psi'_{nj} d\xi \tag{18}$$
$$M_{ij}^{RI_2mn} = \int_0^1 \kappa\psi'_{mi}\psi'_{nj} d\xi \tag{19}$$
$$K_{ij}^{GAmn} = \int_0^1 (1-\xi)\psi'_{mi}\psi'_{nj} d\xi \tag{20}$$
$$K_{ij}^{GBmn} = \int_0^1 \frac{1}{2}(1-\xi^2)\psi'_{mi}\psi'_{nj} d\xi \tag{21}$$
$$K_{ij}^{B_3mn} = \int_0^1 \psi''_{mi}\psi''_{nj} d\xi \tag{22}$$
$$K_{ij}^{B_2mn} = \int_0^1 \kappa\psi''_{mi}\psi''_{nj} d\xi \tag{23}$$
$$K_{ij}^{S^*mn} = \int_0^1 \psi'_{mi}\psi'_{nj} d\xi \tag{24}$$

in which, ψ_{ij} are shape functions of ξ.

The equations of motion (12) - (15) can be written as a matrix form,

$$[M]\left\{\ddot{\theta}\right\} + [K]\{\theta\} = 0 \tag{25}$$

where $[M]$ is the mass matrix and $[K]$ is the stiffness matrix, each of order $(\mu_{2b} + \mu_{2s} + \mu_{3b} + \mu_{3s}) \times (\mu_{2b} + \mu_{2s} + \mu_{3b} + \mu_{3s})$. $\left\{\ddot{\theta}\right\}$ and $\{\theta\}$ are the $(\mu_{2b} + \mu_{2s} + \mu_{3b} + \mu_{3s}) \times 1$ acceleration and displacement vector coordinates, respectively,

$$[M] = \begin{bmatrix} \overline{M}_{11} & \overline{M}_{12} & \overline{0} & \overline{0} \\ \overline{M}_{21} & \overline{M}_{22} & \overline{0} & \overline{0} \\ \overline{0} & \overline{0} & \overline{M}_{33} & \overline{M}_{34} \\ \overline{0} & \overline{0} & \overline{M}_{43} & \overline{M}_{44} \end{bmatrix} \tag{26}$$

$$[K] = \begin{bmatrix} \overline{K}_{11} & \overline{K}_{12} & \overline{0} & \overline{0} \\ \overline{K}_{21} & \overline{K}_{22} & \overline{0} & \overline{0} \\ \overline{0} & \overline{0} & \overline{K}_{33} & \overline{K}_{34} \\ \overline{0} & \overline{0} & \overline{K}_{43} & \overline{K}_{44} \end{bmatrix} \tag{27}$$

$$\{\overline{\theta}\} = \begin{Bmatrix} \overline{\theta}_{2bj} \\ \overline{\theta}_{2sj} \\ \overline{\theta}_{3bj} \\ \overline{\theta}_{3sj} \end{Bmatrix} \tag{28}$$

Each element of $[M]$ in Eq. (26) and of $[K]$ in Eq. (27) is in fact a sub matrix defined as follows.

$$\overline{M}_{11} = \left(M_{ij}^{2b2b} + \frac{1}{\beta} M_{ij}^{RI_3 2b2b} \right) \tag{29}$$

$$\overline{M}_{12} = M_{ij}^{2b2s} \tag{30}$$

$$\overline{M}_{21} = M_{ij}^{2s2b} \tag{31}$$

$$\overline{M}_{22} = M_{ij}^{2s2s} \tag{32}$$

$$\overline{M}_{33} = \left(M_{ij}^{3b3b} + \frac{1}{\beta} M_{ij}^{RI_2 3b3b} \right) \tag{33}$$

$$\overline{M}_{34} = M_{ij}^{3b3s} \tag{34}$$

$$\overline{M}_{43} = M_{ij}^{3s3b} \tag{35}$$

$$\overline{M}_{44} = M_{ij}^{3s3s} \tag{36}$$

$$\overline{K}_{11} = \gamma^2 \left(K_{ij}^{GB2b2b} + \delta K_{ij}^{GA2b2b} - M_{ij}^{2b2b} \right) + K_{ij}^{B_3 2b2b} \tag{37}$$

$$\overline{K}_{12} = \gamma^2 \left(K_{ij}^{GB2b2s} + \delta K_{ij}^{GA2b2s} - M_{ij}^{2b2s} \right) \tag{38}$$

$$\overline{K}_{21} = \gamma^2 \left(K_{ij}^{GB2s2b} + \delta K_{ij}^{GA2s2b} - M_{ij}^{2s2b} \right) \tag{39}$$

$$\overline{K}_{22} = \gamma^2 \left(K_{ij}^{GB2s2s} + \delta K_{ij}^{GA2s2s} - M_{ij}^{2s2s} \right) + \beta \eta K_{ij}^{S^* 2s2s} \tag{40}$$

$$\overline{K}_{33} = \gamma^2 \left(K_{ij}^{GB3b3b} + \delta K_{ij}^{GA3b3b} \right) + K_{ij}^{B_2 3b3b} \tag{41}$$

$$\overline{K}_{34} = \gamma^2 \left(K_{ij}^{GB3b3s} + \delta K_{ij}^{GA3b3s} \right) \tag{42}$$

$$\overline{K}_{43} = \gamma^2 \left(K_{ij}^{GB3s3b} + \delta K_{ij}^{GA3s3b} \right) \tag{43}$$

$$\overline{K}_{44} = \gamma^2 \left(K_{ij}^{GB3s3s} + \delta K_{ij}^{GA3s3s} \right) + \beta \eta K_{ij}^{S^* 3s3s} \tag{44}$$

In the above equations, the M_{ij}'s and K_{ij}'s are actual matrices with elements being defined in Eqs. (17)-(24).

Numerical Example

A rotating beam with $\kappa = 0.5$, $\delta = 1$, $\eta = 0.25$, and $\beta = 1E6$, is analyzed, using the approach proposed in the present study. The locus of the four lowest natural frequencies of a rotating beam is shown in Fig. 1.

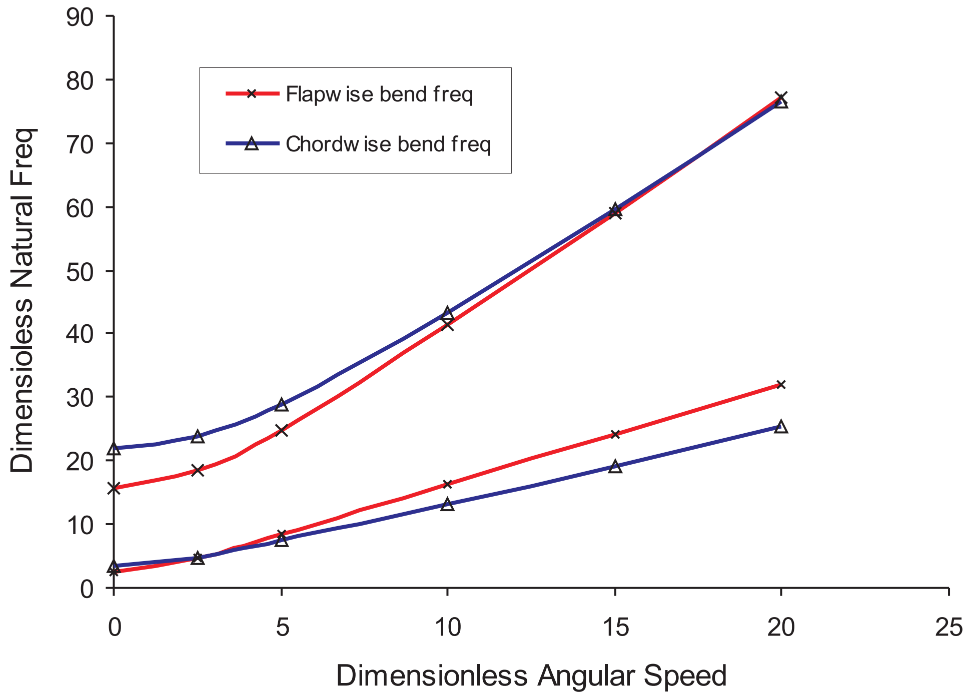

Figure 1: Four lowest natural frequencies ($\kappa = 0.5$, $\delta = 1$, $\eta = 0.25$, and $\beta = 1E6$)

In Fig. 1, veering occurs when γ (angular speed ratio) >2.5 for the first-second frequencies and γ >15 for the third-fourth frequencies.

References

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