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A Nonlinear Meshless Local Petrov-Galerkin (MLPG) Approach Based on the Nonlinear Regular Local Boundary Integral Equation

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Summary

A nonliear meshless local Petrov-Galerkin (NMLPG) method for solving nonlinear boundary value problems, based on the nonlinear regular local boundary integral equation (NRLBIE) and the moving least squares approximation, is proposed in the present paper. No special integration scheme is needed to evaluate the volume and boundary integrals. The integrals in the present method are evaluated only over regularly-shaped sub-domains and their boundaries. This flexibility in choosing the size and the shape of the local sub-domain will lead to a more convenient formulation in dealing with the nonlinear problems. Compared to the original meshless local Petrov-Galerkin (MLPG) method that has been extensively popularized in recent literature, the present method does not need the derivatives of the shape functions in constructing the system stiffness matrix, for those nodes with no displacement specified on their local boundaries. This is very attractive in engineering applications as the calculation of derivatives of the shape functions from the MLS approximation is quite costly. Also compared to the meshless local boundary integral equation (MLBIE) method, the present approach does not involve singular or hyper-singular integrals which have to be tackled in the MLBIE method. Thus, the present method possesses high accuracy, and is cost effective. Numerical examples show that the present method converges fast to the final solution with reasonably accurate results for both the unknown variable and its derivatives.

Introduction

The MLPG methods ([1] – [4]) are effective meshless methods for solving linear and nonlinear boundary value problems. Since the moving least squares (MLS) approximation [1] is used for interpolation in MLPG methods, and the shape functions from the moving least squares approximation are not polynomials, the evaluation of all integrals in the weak form, especially of those integrals involving the derivatives of the shape functions, is very expensive. To overcome this disadvantage, a modified MLPG approach based on a regular local boundary integration equation (RLBIE) was proposed in [5], in which no derivatives of the shape functions are required for internal nodes, thus reducing computational cost. In the present work, the MLPG approach proposed in [5] is extended to solve nonlinear problems.

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The MLPG Based on the Nonlinear Regular Local Boundary Equation

Consider the following nonlinear problem,

$$\nabla^2 u(\mathbf{x}) + \omega^2 u(\mathbf{x}) + \varepsilon u^3(\mathbf{x}) = p(\mathbf{x}) \quad \mathbf{x} \in \Omega \tag{1}$$

with boundary conditions

$$u = \overline{u} \quad \text{on} \quad \Gamma_u \tag{2a}$$

$$\frac{\partial u}{\partial n} \equiv q = \overline{q} \quad \text{on} \quad \Gamma_q \tag{2b}$$

where p is a given source function; ε is a small parameter ($|\varepsilon| \ll 1$), $\overline{u}$ and $\overline{q}$ are the prescribed potential and normal flux, respectively, on the essential boundary Γ_u and on the flux boundary Γ_q, and n is the outward normal direction to the boundary Γ.

A local weak form of the differential equation (1) and the boundary conditions (2), over a local sub-domain Ω_s, can then be written as:

$$\int_{\Omega_s} (\nabla^2 u + \omega^2 u + \varepsilon u^3 - p) v \, \mathrm{d}\Omega - \alpha \int_{\Gamma_{su}} (u - \overline{u}) v \, \mathrm{d}\Gamma = 0 \tag{3}$$

where u is the trial function, v is the test function, $\alpha >> 1$ is a penalty parameter used to impose the essential boundary conditions, and Γ_{su} is a part of the boundary $\partial\Omega_s$ of Ω_s, over which the essential boundary conditions are specified.

Using $(\nabla^2 u)\, v = u_{,ii} v = (u_{,i} v)_{,i} - (u v_{,i})_{,i} + u v_{,ii}$ and the divergence theorem twice yields:

$$\int_{\partial\Omega_s} v \frac{\partial u}{\partial n} \mathrm{d}\Gamma - \int_{\partial\Omega_s} u \frac{\partial v}{\partial n} \mathrm{d}\Gamma + \int_{\Omega_s} (u \nabla^2 v + \omega^2 u v + \varepsilon u^3 v - p v) \, \mathrm{d}\Omega - \alpha \int_{\Gamma_{su}} (u - \overline{u}) v \, \mathrm{d}\Gamma = 0 \tag{4}$$

where $\partial\Omega_s$ is the boundary of Ω_s and n is outward unit normal to the boundary $\partial\Omega_s$.

Imposing the natural boundary condition, $q = \overline{q}$ in equation (4) gives

$$\begin{aligned} &\int_{L_s} q v \, \mathrm{d}\Gamma + \int_{\Gamma_{su}} q v \, \mathrm{d}\Gamma + \int_{\Gamma_{sq}} \overline{q} v \, \mathrm{d}\Gamma - \int_{\partial\Omega_s} u \frac{\partial v}{\partial n} \mathrm{d}\Gamma \\ &\qquad + \int_{\Omega_s} (u \nabla^2 v + \omega^2 u v + \varepsilon u^3 v - p v) \, \mathrm{d}\Omega - \alpha \int_{\Gamma_{su}} (u - \overline{u}) v \, \mathrm{d}\Gamma = 0 \end{aligned} \tag{5}$$

where u is the trial function, v is the test function, and Γ_{su} is a part of the boundary $\partial\Omega_s$ of Ω_s, over which the essential boundary conditions are specified. In general, $\partial\Omega_s = \Gamma_s \cup L_s$, with Γ_s being a part of the local boundary located on the global

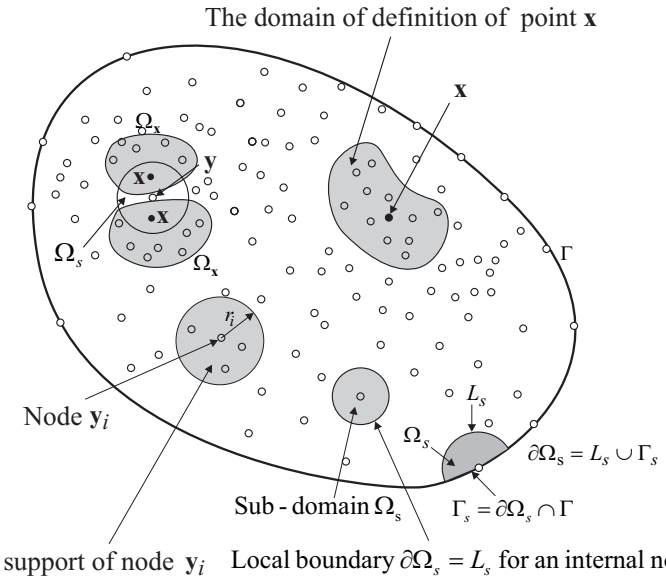

Figure 1: The local domains, the supports of nodes, the domain of definition of the MLS approximation for the trial function at a point, and the domain of influence of a source point (node)

boundary and L_s being the other part of the local boundary over which no boundary condition is specified, i.e., $\Gamma_s = \partial\Omega_s \cup \Gamma$ and $L_s = \partial\Omega_s - \Gamma_s$ (see Fig. 1). If the sub-domain Ω_s is located entirely within the global domain Ω, and there is no intersection between the local boundary $\partial\Omega_s$ and the global boundary Γ, the boundary integral over Γ_{su} vanishes. In Eq (3), a penalty parameter $\alpha >> 1$ is used to impose the essential boundary conditions, as the MLS approximation will be used to approximate the trial function, and it is not easy to directly impose the essential boundary conditions, *a priori*, in the MLS approximation.

The above equation can be further simplified such that the test function v vanishes on L_s. This can be easily accomplished by using the weight function in the MLS approximation as also the test function, with the radius r_i of the support of the weight function being replaced by the radius r_0 of the local domain Ω_s, such that the test function vanishes on a circle of radius r_0. Using this test function and rearranging equation (5), we obtain the following regular local boundary integral equation (RLBIE):

$$\int_{\Gamma_{su}} (q-\alpha u)v\,\mathrm{d}\Gamma - \int_{\partial\Omega_s} u\frac{\partial v}{\partial n}\,\mathrm{d}\Gamma + \int_{\Omega_s} (u\nabla^2 v + \omega^2 uv + \varepsilon u^3 v)\,\mathrm{d}\Omega$$

$$= \int_{\Omega_s} pv\,\mathrm{d}\Omega - \int_{\Gamma_{sq}} \overline{q}v\,\mathrm{d}\Gamma - \alpha \int_{\Gamma_{su}} \overline{u}v\,\mathrm{d}\Gamma\,. \tag{6}$$

In the present formulation, the equilibrium equation and the boundary conditions are satisfied, *a posteriori*, in all local sub-domains and on their Γ_s, respectively. Theoretically, as long as the union of all local domains covers the global domain, i.e., $\cup\Omega_s \supset \Omega$, the equilibrium equation and the boundary conditions will be satisfied, *a posterior*, in the global domain Ω and on its boundary Γ, respectively.

The nonlinear RLBIE (6) can be linearized to solve the increment Δu:

$$\begin{aligned}\int_{\Gamma_{su}} (\Delta q - \alpha\Delta u)v\,\mathrm{d}\Gamma - \int_{\partial\Omega_s} \Delta u\frac{\partial v}{\partial n}\,\mathrm{d}\Gamma + \int_{\Omega_s} (\Delta u\nabla^2 v + \omega^2\Delta uv + 3\varepsilon u^2\Delta uv)\mathrm{d}\Omega \\ = \int_{\Omega_s} (t+\Delta t)\overline{p}v\,\mathrm{d}\Omega - \int_{\Gamma_{sq}} (t+\Delta t)\overline{q}^* v\,\mathrm{d}\Gamma - \alpha\int_{\Gamma_{su}} (t+\Delta t)\overline{u}^* v\,\mathrm{d}\Gamma \\ - \int_{\Gamma_{su}} (q-\alpha u)v\,\mathrm{d}\Gamma + \int_{\partial\Omega_s} u\frac{\partial v}{\partial n}\,\mathrm{d}\Gamma - \int_{\Omega_s} (u\nabla^2 v + \omega^2 uv + \varepsilon u^3 v)\mathrm{d}\Omega\end{aligned} \tag{7}$$

where $\overline{p}$, $\overline{u}^*$, and $\overline{q}^*$ are define as

$$p = t\overline{p};\quad \overline{u} = t\overline{u}^*;\quad \overline{q} = t\overline{q}^*;\quad 0 \le t \le 1\,. \tag{8}$$

To obtain the discrete equations from the nonlinear RLBIE (7), the MLS approximation (See [1] for details of the MLS approximation), $u(\mathbf{x}) = \sum_{i=1}^{n} \phi_i(\mathbf{x})\hat{u}_i$, is used to approximate the trial function u. Using the MLS approximation into the nonlinear RLBIE (7) for all nodes leads to the following discretized system of linear equations:

$$\mathbf{K}\cdot\Delta\hat{\mathbf{u}} = \mathbf{f} \tag{9}$$

where, the entries of the stiffness matrix $\mathbf{K}$ and the load vector $\mathbf{f}$ are define by

$$\begin{aligned}K_{ij} = \int_{\Gamma_{su}} [\phi_{j,n}(\mathbf{x}) - \alpha\phi_j(\mathbf{x})]v(\mathbf{x}_i,\mathbf{x})\,\mathrm{d}\Gamma - \int_{\partial\Omega_s} \phi_j(\mathbf{x})\frac{\partial v(\mathbf{x}_i,\mathbf{x})}{\partial n}\,\mathrm{d}\Gamma \\ + \int_{\Omega_s} [\phi_j(\mathbf{x})\nabla^2 v(\mathbf{x}_i,\mathbf{x}) + \omega^2\phi_j(\mathbf{x})v(\mathbf{x}_i,\mathbf{x}) + 3\varepsilon u^2(\mathbf{x})\phi_j(\mathbf{x})v(\mathbf{x}_i,\mathbf{x})]\mathrm{d}\Omega\end{aligned} \tag{10a}$$

and

$$\begin{aligned}f_i = \int_{\Omega_s} (t+\Delta t)\overline{p}(\mathbf{x})v(\mathbf{x}_i,\mathbf{x})\,\mathrm{d}\Omega - \int_{\Gamma_{sq}} (t+\Delta t)\overline{q}^*(\mathbf{x})v(\mathbf{x}_i,\mathbf{x})\,\mathrm{d}\Gamma \\ - \alpha\int_{\Gamma_{su}} (t+\Delta t)\overline{u}^*(\mathbf{x})v(\mathbf{x}_i,\mathbf{x})\,\mathrm{d}\Gamma - \int_{\Gamma_{su}} (q(\mathbf{x}) - \alpha u(\mathbf{x}))v(\mathbf{x}_i,\mathbf{x})\,\mathrm{d}\Gamma \\ + \int_{\partial\Omega_s} u(\mathbf{x})\frac{\partial v(\mathbf{x}_i,\mathbf{x})}{\partial n}\,\mathrm{d}\Gamma\end{aligned}$$

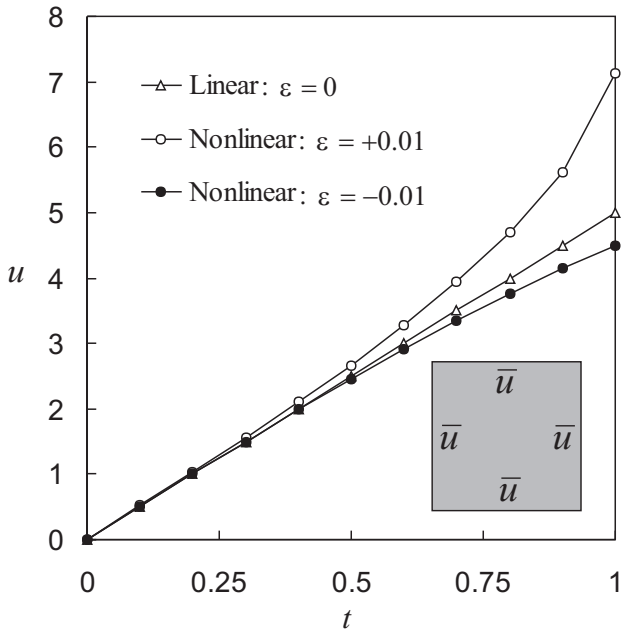

Figure 2: The hardening and softening nonlinearities

$$-\int_{\Omega_s}[u(\mathbf{x})\nabla^2 v(\mathbf{x}_i,\mathbf{x})+\omega^2 u(\mathbf{x})v(\mathbf{x}_i,\mathbf{x})+\varepsilon u^3(\mathbf{x})v(\mathbf{x}_i,\mathbf{x})]\mathrm{d}\Omega \tag{10b}$$

in which ϕ_j is the shape function from the MLS approximation, and $v(\mathbf{x},\mathbf{x}_i)$ denotes the test function corresponding to the node at $\mathbf{x}_i$.

It is seen from Eq. (10a) that no derivatives of the shape functions are needed in constructing the stiffness matrix for the internal nodes and for those boundary nodes with no essential-boundary-condition-prescribed sections on their local boundaries. This is attractive in engineering applications as the calculation of derivatives of the shape functions from the MLS approximation is quite costly.

Numerical Example

We consider a problem defined over the domain $\pi \times \pi$, with $\overline{u} = 0$ specified on all sides and the source function p being given by

$$p(\mathbf{x}) = 5t\ \sin x_1\ \sin x_2 \tag{11}$$

in which t is the load parameter with $0 \le t \le 1$. Of course, the exact solution is not available unless when $\varepsilon = 0$ — the linear problem.

A regular mesh with 36 nodes is tested in this problem. The Gaussian weight function and quadratic basis (see [1]) are used in the computation. The constant ε

is taken to be 0.01 and -0.01 in the computation to verify the "hardening" and "softening" nonlinearities. The values of u at the middle point $(x_1,\ x_2) = (\pi/2,\ \pi/2)$ are computed for different t, and sketched in Figure 2. Figure 2 clearly shows the hardening and softening nonlinearities of the problem, for $\varepsilon = 0.01$ and $\varepsilon = -0.01$, respectively.

Conclusions and Discussions

Compared with other meshless techniques based on a global weak form, the present approach does not require a finite element mesh for interpolation purposes or for integration purposes, while shadow elements are required to evaluate volume integrals in meshless methods based on a global weak form. Compared with other meshless methods based on a local weak form, the present method does not involve singular or hyper-singular integrals that are inevitable in the MLBIE method, the present approach is not a subdomain collocation method, and the present method does not need the derivatives of the shape functions in the domain integrals, which are inevitable in the original MLPG. It is seen from the implementation that no derivatives of the shape functions are required in constructing the stiffness matrix for those nodes with no displacement boundary condition specified on their local boundaries. This is attractive in engineering applications as the calculation of derivatives of the shape functions from the MLS approximation is quite costly.

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