MODAL IDENTIFICATION OF AN AEROELASTIC SYSTEM USING AN EXTENDED KARHUNEN-LOÈVE DECOMPOSITION

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ABSTRACT

The paper presents recent applications of an output-only technique for modal identification of systems with non-uniform mass, based on an extension of the Karhunen-Loève Decomposition (KLD). The method is here applied to identify the aeroelastic modes of a wing with a concentrated mass in a uniform flow. First, the spatial modal shapes of the coupled system are evaluated as the eigenfunctions (eigenvectors in the numerical approach) of the so-called extended Karhunen-Loève integral operator, whose L^2*-kernel is the time-averaged autocorrelation tensor of the elastic displacement vector of the wing in the flow (available from experiments or computer simulations), multiplied by the density function of the structure. Then, the identification of the aeroelastic modal parameters is completed by considering the projection of the elastic displacement vector onto the Karhunen-Loève eigenfunctions. Frequency and damping associated to each aeroelastic mode are evaluated as the solution of a multi-dimensional minimization problem, based on the optimal matching of the projection with an ideal damped oscillator. The methodology is here validated on the basis of a computer simulation and different approaches are shown. The output modes are in a very good agreement with the aeroelastic modes used to build the numerical input. Frequency and damping of each mode are also in a good agreement with the relative input values.*

1. INTRODUCTION

The Karhunen-Loève Decomposition is a statistical method for finding a base that cover the optimal distribution of energy in the dynamics of a continuum.

This method initially appeared in the signal processing literature, where it was presented by Hotelling (1933) as the Principal Component Analysis (PCA). The theory behind the method was taken again and studied in depth by Kosambi (1943), by Loève (1945) and by Karhunen (1946). Since it was applied by Lumley (1967) to uncover coherent structures in turbulent flows, it has become a standard tool in turbulence studies (Holmes, 1996), where it is also known as the Proper Orthogonal Decomposition (POD).

The theory proposed by Karhunen (1946) and Loève (1945) is recently emerging as a powerful tool in structural dynamics and vibration. A physical interpretation of the use of the KLD in vibrations studies has been shown by Feeny et al (1998). In structural dynamics, the method consists in constructing the time-averaged spatial autocorrelation tensor of the elastic displacement field of the structure. Its spectral analysis produces a basis, as a set of orthonormal eigenfunctions (eigenvectors, in the numerical approach) with the corresponding set of eigenvalues, which represent the energy content of each mode. It has been shown (Feeny et al, 1998) that for undamped and unforced structures with constant density, the eigenfunctions given by the standard KLD coincide with the natural modes of vibration. Recently, the formulation has been extended by Iemma et al (2006a) to the modal identification of structures with non-uniform density. It is worth noting that this extension of the KLD may be applied to the modal analysis of n-dimensional structures ($n = 1, 2, 3$).

In aeroelasicity studies, POD techniques have been widely used for reduced order models (ROMs) determination. The reduction of aeroelastic equations via KL basis has been shown, e.g., by Romanowsky (1996). The technique has been extended to nonlinear aeroelasticity, allowing the identification of both aerodynamic and aeroelastic KL-based ROMs (Pettit and Beran, 2000; Lucia et al, 2003).

In this work, the technique presented in Iemma et al (2006a) is applied to the modal identification of an aeroelastic system. Specifically, a wing with a concentrated mass in a uniform flow is analyzed. Different approaches and methods are shown and discussed. A method for estimating frequency, damping and amplitude of the complex vibration is also shown. In the next sections, the general theory underlying the Karhunen-Loève decomposition is recalled, with emphasis on its application to quasi-periodic dynamical systems with non-uniform density. The extension of the KLD to non-uniform density structures is briefly outlined. The method for estimating the relevant modal parameters is also shown and the results, based on numerical experiments, are presented.

2. EXTENDED KARHUNEN-LOÈVE DECOMPOSITION

In this section we briefly outline the general theory underlying the Karhunen-Loève decomposition with its extension to non-uniform density structures. For the sake of simplicity, we recall the formulation as apply for structural dynamics. At the end of the section, the problem will be extended to a coupled system in aeroelasticity.

In structural dynamics, the method introduced by Karhunen and Loève is used to provide a basis for the *optimal* representation of the displacement vector $\mathbf{u}(\mathbf{x},t)$ of a vibrating inhomogeneous structure. The method provides a basis which is optimal, in the energy content sense, for the representation of the displacement vector $\mathbf{u}(\mathbf{x},t)$ in the linear combination $\mathbf{u}(\mathbf{x},t) = \sum_{k=1}^{n} \beta_k(t)\, \boldsymbol{\varphi}_k(\mathbf{x})$, truncated to the order n, with $\mathbf{x} \in \mathcal{D}$ and $t \in [0,T]$.[1] The optimality condition associated to the KLD ensures that, for a given n, the first n KLD basis functions capture, on average, more energy than any other orthonormal basis in the linear representation of the field $\mathbf{u}$ (Holmes, 1996). It has been shown that this property is satisfied (under certain conditions) by the natural modes, provided that the formulation is embedded in the proper Hilbert space (Iemma et al, 2006a). In the following, the theory underlying the extension of the KLD to the modal identification of inhomogeneous structures is briefly recalled.

We assume that the dynamics of the undamped-unforced structure is governed by the equation $\rho(\mathbf{x})\ \ddot{\mathbf{u}}(\mathbf{x},t) + \mathcal{L}\,\mathbf{u}(\mathbf{x},t) = 0$, where $\rho = \rho(\mathbf{x})$ is the structure density. Thus, the displacement vector is given by $\mathbf{u}(\mathbf{x},t) = \sum_{k=1}^{\infty} \alpha_k(t)\ \boldsymbol{\phi}_k(\mathbf{x})$, where $\boldsymbol{\phi}_k(\mathbf{x})$ are the natural modes of vibration (linear normal modes), solution of $\mathcal{L}\,\boldsymbol{\phi}_k(\mathbf{x}) = \rho(\mathbf{x})\,\mu_k\,\boldsymbol{\phi}_k(\mathbf{x})$, with $\int_{\mathcal{D}} \rho(\mathbf{x})\,\boldsymbol{\phi}_i(\mathbf{x}) \cdot \boldsymbol{\phi}_j(\mathbf{x})\,\mathrm{d}\mathbf{x} = \delta_{ij}$. The time dependency of the solution is given by $\alpha_k(t) = a_k\ \cos(\omega_k t + \chi_k)$, where $\omega_k = \sqrt{\mu_k}$, and a_k, $\chi_k \in \Re$.

Assuming that the displacement vector (at a given time) belongs to the Hilbert space $L^2_\rho(\mathcal{D})$, defined by the inner product $(\mathbf{f},\mathbf{g})_\rho := \int_{\mathcal{D}} \rho(\mathbf{x})\,\mathbf{f}(\mathbf{x})\cdot\mathbf{g}(\mathbf{x})\,\mathrm{d}\mathbf{x}$, the optimal decomposition of the vector $\mathbf{u}$ is given by the solutions of the integral problem (a complete proof of the following equation is given by Iemma et al (2006a) and, thus, not repeated here)

$$\mathcal{L}^E_{\mathbf{R}}\varphi(\mathbf{x}) := \int_{\mathcal{D}} \rho(\mathbf{y})\mathbf{R}(\mathbf{x},\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \lambda\boldsymbol{\varphi}(\mathbf{x}), \tag{1}$$

where $\mathbf{R}(\mathbf{x},\mathbf{y}) := \langle \mathbf{u}(\mathbf{x},t) \otimes \mathbf{u}(\mathbf{y},t)\rangle$ is the time-averaged autocorrelation tensor of the displacement vector $\mathbf{u}(\mathbf{x},t)$, being $\langle\ldots\rangle := \int_0^T \ldots \mathrm{d}t$ the time-averaging operator and $\otimes$ the standard tensor product. $\mathcal{L}^E_{\mathbf{R}}$ is the extended Karhunen-Loève integral operator and the KLD optimal basis is given by its eigensolutions. It may be shown that $\mathcal{L}^E_{\mathbf{R}}$ is selfadjoint in $L^2_\rho(\mathcal{D})$, *i.e.*, $(\mathbf{f}, \mathcal{L}^E_{\mathbf{R}}\,\mathbf{g})_\rho = (\mathcal{L}^E_{\mathbf{R}}\,\mathbf{f}, \mathbf{g})_\rho$, and compact (since the kernel of Equation 1 is bounded). Hence, its eigenvalues are real and its eigenfunctions form a complete set of orthogonal functions in the above-defined Hilbert space (Kress, 1989). Under the hypothesis of undergoing unforced free vibrations and assuming an observation time T tending to infinity, the Karhunen-Loève eigenfunctions coincide with the natural modes of the structure, *i.e.*, $\boldsymbol{\varphi}_k(\mathbf{x}) = \boldsymbol{\phi}_k(\mathbf{x})$ and, in addition, $\lambda_k = \frac{1}{2}\ a_k^2$ (Iemma et al, 2006a). In practical applications, proper modal identification has to be expected if the observed vibration is representative of the motion from a statistical point of view. This is ensured if all the modes present in the motion undergo a sufficient number of periods during the acquisition time. Thus, the acquisition time has to be sufficiently long provided, of course, that damping is not too high.

When a structure (such as a wing) is embedded in a uniform flow, the dynamics may be written as

$$\rho(\mathbf{x})\ \ddot{\mathbf{u}}(\mathbf{x},t) + \mathcal{L}\,\mathbf{u}(\mathbf{x},t) = \mathbf{f}_A(\ddot{\mathbf{u}},\dot{\mathbf{u}},\mathbf{u},\mathbf{x},t) \tag{2}$$

In most application of interest in aeronautics, the dependence of the aerodynamic forces on the elastic displacements (and their time-derivatives) may be assumed as linear (Iemma

[1] Note that, in general, $\mathbf{x} \in \mathbb{E}^n, n = 1,2,3$ and $\mathbf{u}(\mathbf{x},t) \in \mathbb{V}^m, m = 1,2,3$, being $\mathbb{E}^n$ an n-dimensional ($n = 1,2,3$) Euclidean point space and $\mathbb{V}^m$ an m-dimensional ($m = 1,2,3$) vector space, with n not necessarily equal to m; consider, for instance, the case of a bending beam ($n = 1, m = 2$), or of a bending plate ($n = 2, m = 1$).

and Gennaretti, 2005). Nevertheless, the resulting aeroelastic operator is not self-adjoint and the aeroelastic modes for the representation of $\mathbf{u}$ are not orthogonal in the $L^2_\rho(\mathcal{D})$ space. In this case, the KLD eigenfunctions (always orthogonal in the embedding Hilbert space) are expected to be a good approximation of those aeroelastic modes that are "quasi" orthogonal (*i.e.*, for which $\int_\mathcal{D} \rho\, \boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_j \mathrm{d}\mathbf{x} = \epsilon_{ij}, i \neq j$). In other words, proper modal identification is expected in the subspace spanned by "quasi" orthogonal aeroelastic modes.

3. FREQUENCY, DAMPING AND AMPLITUDE ESTIMATE

Frequencies, damping and modal amplitudes are evaluated using the following technique. First, the coefficients $\beta_k(t)$ (see previous Section) are computed as the $L^2_\rho(\mathcal{D})$-projections of the vector $\mathbf{u}(\mathbf{x}, t)$ onto the k-th Karhunen-Loève mode, *i.e.*, $\beta_k(t) = (\mathbf{u}, \boldsymbol{\varphi}_k)_\rho := \int_\mathcal{D} \mathbf{u}(\mathbf{x}, t) \cdot \boldsymbol{\varphi}(\mathbf{x})\mathrm{d}\mathbf{x}$. Then the above coefficients are Fourier-transformed. Under the hypothesis of proper modal identification (*i.e.*, under the hypothesis that $\boldsymbol{\phi}_k = \boldsymbol{\varphi}_k$) the frequency associated to the k-th Karhunen-Loève mode is evaluated and assumed as the frequency associated to the corresponding vibrational mode (Iemma et al, 2006b).

The estimation of damping is conducted by approximating the k-th coefficient $\beta_k(t)$ with the ideal damped oscillator (solution of $\ddot{\beta} + 2\gamma\dot{\beta} + k\beta = 0$, with $\beta(0) = \beta_0$ and $\dot{\beta}(0) = \beta_1$):

$$\hat{\beta}_k(t) := \hat{a}_k e^{-\hat{\gamma}_k t} \sin(\hat{\omega}_k t + \hat{\chi}_k) \tag{3}$$

(being $2\hat{\gamma}_k$ the damping associated to the k-th mode, $\hat{\chi}_k$ the phase and $\hat{a}_k$ the amplitude) and finding those parameters $\hat{a}_k, \hat{\gamma}_k, \hat{\omega}_k, \hat{\chi}_k$ that solve the problem

$$\underset{\hat{a}_k, \hat{\gamma}_k, \hat{\omega}_k, \hat{\chi}_k}{\text{Minimize}} \quad \int_0^T |\hat{\beta}_k(t) - \beta_k(t)|^2 \mathrm{d}t. \tag{4}$$

4. NUMERICAL RESULTS AND DISCUSSION

In this section, results obtained on the basis of numerical experiments are presented. We apply the modal identification procedure to a wing with a concentrated mass of 10,000 kg located at the tip section, embedded in a uniform flow. The lifting structure has a mass of 5,625 kg. The wing plant is depicted in Fig. 1. The wing section, constant along the span, is a circular biconvex airfoil with $t/c = 0.02$. In the following, we first outline the method used to build the numerical input; then we solve the KLD problem in both standard and extended formulation, and compare the results. Finally, in order to overcome the non-orthogonality of the aeroelastic modes, we apply the KLD to a properly filtered signal and perform the time modeling.

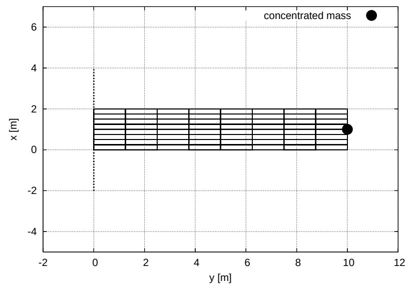

Figure 1: *Wingplant (semi-span).*

4.1. Building the input

In order to build the numerical input of the KLD procedure, we solve the problem of Eq. 2 using a Galerkin expansion of the type $\mathbf{u}(\mathbf{x}, t) \approx \sum_{k=1}^N q_k(t)\, \boldsymbol{\psi}_k(\mathbf{x})$. For the $\boldsymbol{\psi}_k$ we use the natural modes of vibration of the structure in vacuum as computed by a finite elements method. The aerodynamic loads are evaluated via a boundary element method, BEM (Iemma and Gennaretti, 2005), and coupled with the structural dynamics with a finite state approximation technique (Iemma and Gennaretti, 2005).

In this work, we use $N = 3$, and the we assume a solution of the type

$$\hat{\mathbf{u}}(\mathbf{x}, t) = \sum_{h=1}^{3} c_h \sum_{k=1}^{3} [\tilde{q}_k^{(h)} e^{s_h t} + \tilde{q}_k^{*(h)} e^{s_h^* t}] \boldsymbol{\psi}_k(\mathbf{x}) \tag{5}$$

where * denotes the complex conjugate; s_h are the solutions in the Laplace domain and c_h are arbitrary initial conditions. In the following, we will refer to the s_h as the aeroelastic eigenvalues and to the $\mathbf{q}^{(h)}$ as the aeroelastic eigenvectors. Since the KLD is expected to be independent on the phase shift of the eigenvectors components,[2] the h-th (real) combination $\sum_{k=1}^3 |\tilde{q}_k^{(h)}| \boldsymbol{\psi}_k(\mathbf{x}) := \boldsymbol{\phi}_h(\mathbf{x})$ is used as an input reference function for the h-th aeroelastic mode

[2]The issue would deserve a careful discussion and will be addressed in future work.

h	s_h (s_h^*)
1	$-0.2857 \pm 10.2523i$
2	$-0.2418 \pm 7.9514i$
3	$-0.0215 \pm 0.5685i$

Table 1: *Aeroelastic eigenvalues*

$\tilde{\mathbf{q}}^{(1)}$ $[\tilde{\mathbf{q}}^{*(1)}]$	$\tilde{\mathbf{q}}^{(2)}$ $[\tilde{\mathbf{q}}^{*(2)}]$	$\tilde{\mathbf{q}}^{(3)}$ $[\tilde{\mathbf{q}}^{*(3)}]$
$-0.0003 \mp 0.0047i$	$0.0498 \mp 0.0217i$	$0.9670 \pm 0.2546i$
$-0.0196 \mp 0.0098i$	$-0.3883 \pm 0.4016i$	$0.0001 \mp 0.0006i$
$-0.7827 \pm 0.6223i$	$-0.7866 \pm 0.3210i$	$0.0008 \mp 0.0023i$

Table 2: *Aeroelastic eigenvectors*

(see Fig. 2), for later comparisons. The three sets of aeroelastic complex eigenvalues and eigenvectors are shown in Tabs. 1 and 2. Comparing Eqs. 3 and 5 it is apparent that, under the hypothesis of proper modal identification, must be $-\hat{\gamma}_h = Real(s_h)$ and $\hat{\omega}_h = Imag(s_h)$. The procedure for finding the values of $\hat{\gamma}_h$ and $\hat{\omega}_h$ will be validated in the following on the basis of the known input values. In real-life applications this procedure will give the real and imaginary part of the unknown couple of complex conjugate aeroelastic eigenvalues, associated to the h-th mode $\boldsymbol{\phi}_h$.

4.2. KLD procedure

First, we process the field $\mathbf{u}(\mathbf{x}, t)$ with the standard KLD. Specifically we use a 9x9 nodes grid for the spatial resolution and we process 10,000 time samples at 100 Hz. Second, we use the extension of the KLD to non-uniform structures for the same analysis. The results are shown in Figs. 3 and 4. Figure 5 shows a comparison on a $x = c/4 = 0.5$m cut of the structure between the KL modes and the corresponding aeroelastic input modes. It may be noted how the extended KLD is able to identify the modal shapes of "quasi" orthogonal input modes (as the aeroelastic mode #1 and #3, see Tab. 2). As mentioned, the extended KLD may be used for proper modal identification in the subspace spanned by "quasi" orthogonal input modes.

In order to overcome this limitation, we process the signal associated to a single input mode (through a suitable signal filtering). This technique is closely related to the so-called frequency domain decomposition and may be used to identify the whole set of input modes. The results are shown in Fig 6 where a comparison with the input modes is made on a $x = c/4 = 0.5$m cut of the wing. The agreement is remarkable. The latter results are used to perform the analysis of frequency, damping and amplitude associated to each mode. The projections of the input signal are compared to an ideal damped oscillator and the minimization problem of Eq. 4 is solved. The results are shown in Tab. 3 for KL mode #2 and compared with the input values. Figure 7 depicts a comparison between the projection for KL mode #2 and the corresponding solution for the ideal damped oscillator, showing a very good agreement. Moreover, the amplitude $\hat{a}_h$ (or, in alternative, the h-th KL eigenvalue) may be assumed as an indicator of the modal activity in the vibration observed.

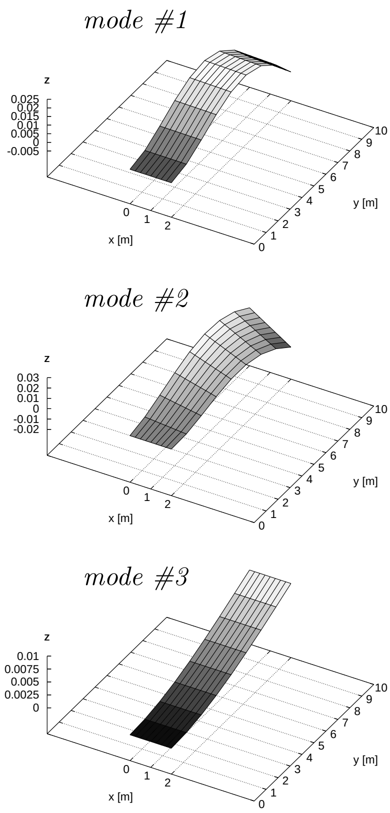

Figure 2: *Input aeroelastic reference modes.*

	$Real(s_h)$	$Imag(s_h)$
input values	-0.241826	7.951417
output values	-0.241826	7.951419

Table 3: *Identification of aeroel. eigenvalue #2*

5. CONCLUSIONS

An output-only technique for modal identification of aeroelastic systems with non-uniform mass, based on an extended Karhunen-Loève De-

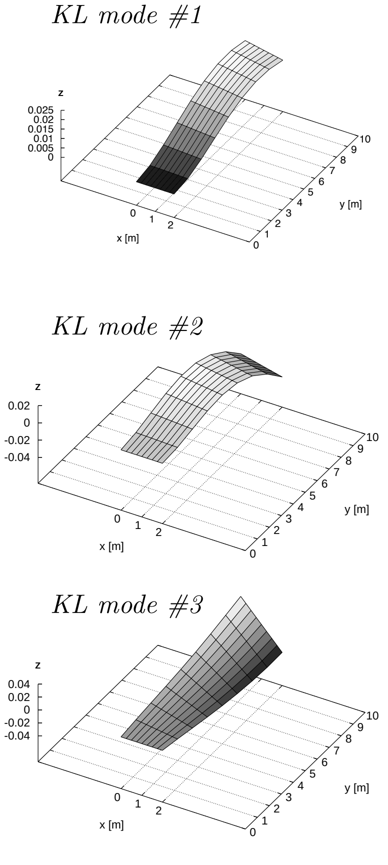

Figure 3: *First three KL modes (standard KLD).*

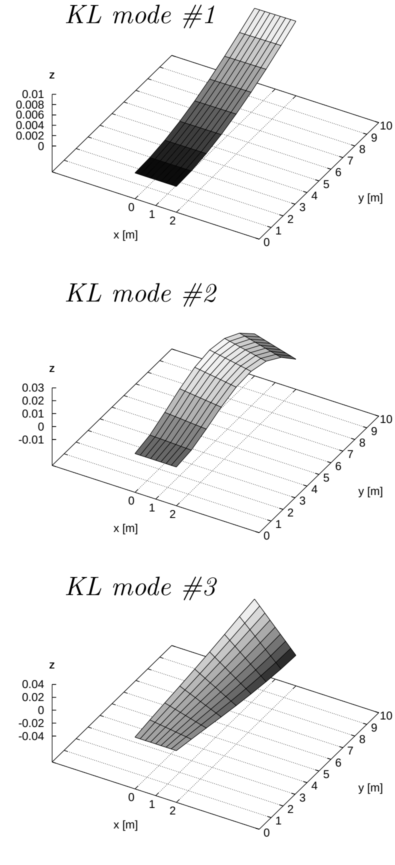

Figure 4: *First three KL modes (extended KLD).*

composition has been shown. Proper modal identification has to be expected for "quasi" orthogonal aeroelastic modes. To overcome this limitation, a methodology based on signal filtering has been used and validated. A procedure for the identification of the aeroelastic eigenvalues has been also presented.

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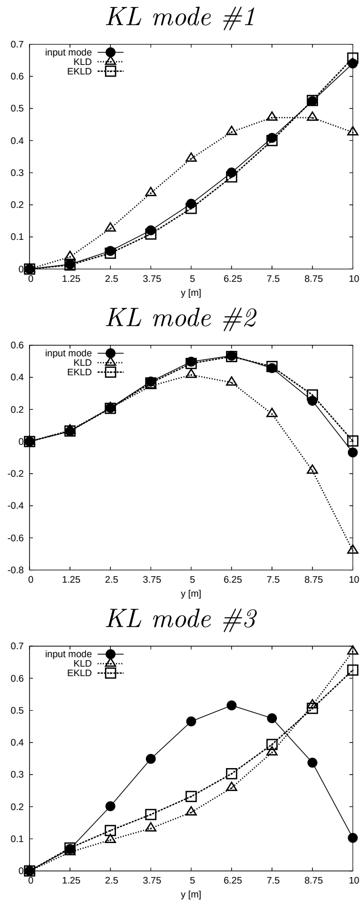

Figure 5: Comparison on a $x = c/4 = 0.5m$ cut of the wing (standard and extended KLD).

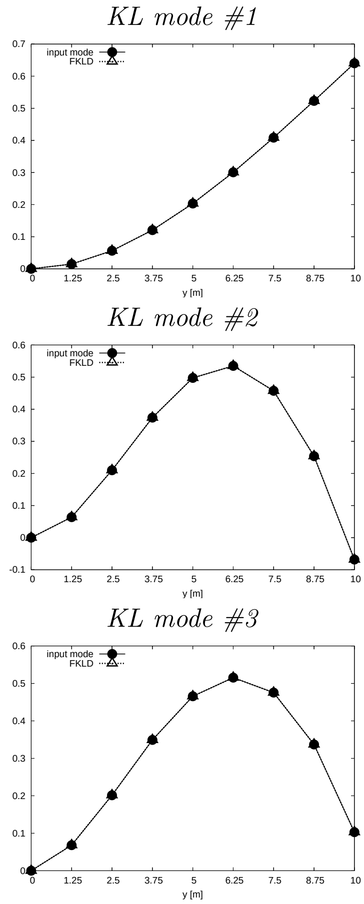

Figure 6: Comparison on a $x = c/4 = 0.5m$ cut of the wing (filtered-signal KLD).

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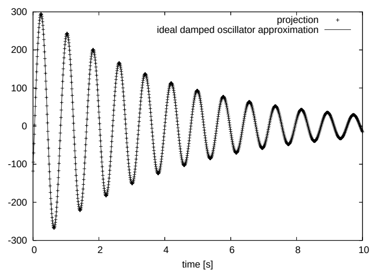

Figure 7: *Ideal damped oscillator approximation for KL mode #2.*