POST-CRITICAL LIMIT CYCLES AND NON-STATIONARY RESPONSE TYPES OF AEROELASTIC SYSTEM

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ABSTRACT

Finding the stationary points and determining stability characteristics of a system is an important undertaking in any attempt to understand or to modify its dynamical properties. The importance can be understood and demonstrated with an example describing the aeroelastic system with gyroscopic as well as with non-conservative terms. We consider a non-linear system with two degrees of freedom representing an interaction of bending and torsion of a slender beam vibrating in a cross flow. The cross-section shape makes possible to separate principal effects and the coupling of aeroelastic modes is caused solely by the flow around the structure. The system is auto-parametric and permits a semi-trivial solution of two types, one of which being unstable. Apart from that the response may be non-trivial in both phase coordinates.

Conditions of existence of stationary points and their types are investigated using primarily the Lyapunov function. The procedure presented is applicable for Hamiltonian, holonomic systems which are conservative, or non-conservative with certain limitations on the generalised forces. The singular points are classified with respect to their asymptotically stable/unstable character together with adequate physical interpretation. Attention is paid to attractive and repulsive areas surrounding these points. Using this background several types of post-critical response in non-linear formulation will be presented, such as stable/unstable limit cycles with various ratio of amplitudes of both components, or quasi-periodic response processes having a form of symmetric or asymmetric beating effects with strong energy trans-flux between degrees of freedom.

1. INTRODUCTION

The most fundamental task of bridge design with regards to the aero-elasticity lies in the formulation of the self-excited forces and interaction with the response. From the theoretical point of view, this interaction leads to the origin of non-conservative forces contributing to the stiffness matrix and in the same time to the origin of forces influencing a damping matrix in linear and non-linear way.

Various levels of the non-linearity strongly influence internal mechanism of self-excited oscillations. On the linear (zero) level as a particular case, two parallel ways, each of them bringing some advantages, can be formulated. The duality of time and frequency domain formulation of the self-excited wind forces was and still is investigated, see e.g. Caracoglia and Jones (2003). In the time domain formulation of self-excited forces on a bridge deck, indicial functions are usually adopted, see e.g. Costa (2007) and others. Aeroelastic problems including the modelling of aerodynamic forces excited by non-stationary wind fields and considering of non-linearities in both structural dynamics and aerodynamics are summarized in Chen and Kareem (2003). For the purpose of direct determination of critical velocities in engineering computations the "combined time/frequency" system can be used despite the iterative methods are used. From mathematical point of view, however, such procedure is considerably problematic, as it implies too many assumptions which probably will not be complied with as soon as the response has lost the character of a purely harmonic movement with clearly expressed frequency, see e.g. Náprstek (2000).

The research has succeeded in understanding most principal characteristics of aeroelastic systems. Although some approaches are able to predict some lower limits of aeroelastic stability loss, they avoid any possibility to investigate the post-critical behaviour which is of strongly non-linear character. However, the detailed knowledge of the post-critical state, is very important being decisive from the viewpoint of a possible secondary restabilisation due to non-linear effects. A number of partial phenomena resulting from the existence of parametric noises and asymmetry of the stiffness matrix (non-conservative forces) and of the damping matrix (gyroscopic forces) have been described successfully in a purely theoretical way, Náprstek (2001).

Besides the pure theory, there has been collected a large number of experimental works related to this topic, see e.g. Ricciardelli et al. (2002). Any of these branches creates its own experimental conditions according to their needs with emphasis on their typical parameters. A systematic research is needed to clarify the role of main individual parameters in the instability onset. It seems that in the given situation it is realistic to start with the development of a uniform theory based simultaneously on experiments which would cover all known cases of aeroelastic instability and possible restabilisation due to non-linear and non-symmetrical terms (vortex shedding incl. noises, galloping, flutter). Some of these phenomena may intermingle and generate further state not yet described theoretically.

2. MODEL OF THE STRUCTURE AND BASIC PROPERTIES OF THE SYSTEM

We refere to the vibration of a slender prismatic bar in the combination of bending and torsion, according to the Fig. 1, circumvent by the wind along its longer side. The form is symmetrical along two axes, its center of torsion is identical with its centroid and the effects of aerodynamic forces can be referred to the center of this section. The coupling of aeroelastic modes is caused solely by the flow around the structure which produce the running lift $L(t)$ and moment $M(t)$.

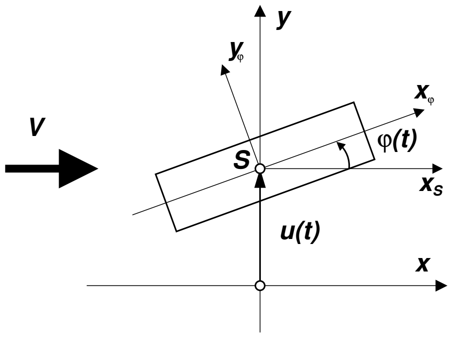

Figure 1. *Schematic depiction of the model with two degrees-of-freedom in the air stream.*

The problem to determine the aeroelastic forces (coefficients) has been dealt with in great detail by some monographs (newly see e.g. Strømmen (2006)), in which the relation between the forces and the air flow velocity has been described by linear formulae. The aerodynamic coefficients are functions of reduced frequency $\kappa = d \cdot \omega / V$ to respect the fact emerging from the physical nature, that they are not static values.

If the difference between the first bending and torsional natural frequencies is small, than it is possible to deduce from practical observations and an energy content assessment that flutter-type vibrations can be described usually by a single natural mode in each component. In such a case for the purpose of qualitative analysis the real beam can be replaced by a two degrees of freedom system the stiffness matrix of which is so defined as to make its natural frequencies corresponding with the respective natural frequencies of the original beam.

For the movement of the system we adopt following general description by a couple of differential equations, see e.g. Náprstek et al. (2007):

$$
\begin{aligned}
&\ddot{u} - 2\omega_{bu}\dot{u} + \omega_u^2 u = \\
&\quad K_m(1 - \gamma_{uu}\dot{u}^2 - \gamma_{u\varphi}\dot{\varphi}^2)(b_{uu}\dot{u} + b_{u\varphi}\dot{\varphi}) + \qquad \text{(a)} \\
&\quad + K_m(1 - \beta_{uu}u^2 - \beta_{u\varphi}\varphi^2)(c_{uu}u + c_{u\varphi}\varphi) \\
&\ddot{\varphi} - 2\omega_{b\varphi}\dot{\varphi} + \omega_\varphi^2 \varphi = \\
&K_J(1 - \gamma_{\varphi u}\dot{u}^2 - \gamma_{\varphi\varphi}\dot{\varphi}^2)(-b_{\varphi u}\dot{u} + b_{\varphi\varphi}\dot{\varphi}) + \qquad \text{(b)} \\
&+ K_J(1 - \beta_{\varphi u}u^2 - \beta_{\varphi\varphi}\varphi^2)(-c_{\varphi u}u + c_{\varphi\varphi}\varphi)
\end{aligned} \tag{1}
$$

where the parameters are: ω_u - eigenfrequency of the vertical motion, ω_φ - rotational eigenfrequency, $\omega_{bu}, \omega_{b\varphi}$ - damping ratios. The coefficients β_{ij} and γ_{ij} represent scaling factors of the non-linear part and form square asymmetric matrices. The parameters c_{ij}, b_{ij} form square asymmetric matrices as well. The coefficients c_{ij}, b_{ij} could be considered as generalizations of very well-known aeroelastic derivatives and should be identified experimentally. Coefficients K_J, K_m depend on the cross-section geometry and the characteristics of the wind:

$$
K_m = \frac{1}{2m}\varrho V^2 \cdot 2d \quad ; \quad K_J = \frac{1}{2J}\varrho V^2 \cdot 2d^2 \tag{2}
$$

with parameters: V - wind velocity, d - characteristic dimension of the beam cross section, ϱ - air density under typical condition.

Roughly speaking the model (1) is oriented on the non-linearities which are described by velocities and displacements. In this meaning they can be compared to the equation of motion of a single-degree-of freedom system with non-linear stiffness part (Duffing), combined with the equation where non-linear damping depends on the square of motion velocity (Rayleigh) Consequently, we name the system (1) Rayleigh-Duffing. Introducing external excitation to the right hand side of the Eq. (1a), the component different from zero will be the displacement $u(t)$ while the rotation $\varphi(t)$ may identically equal zero under certain conditions. Similar result will be attained introducing external excitation only to the right hand side of the Eq. (1b).

Conclusions about the stability of a critical point may be acquired by means of construction of a suitable auxiliary function, called Lyapunov function Φ. Its total time derivative Ψ, can be identified as the rate of change of Φ along the trajectory of the system that passes through the point of interest in the phase plane. There are no general methods of construction of Lyapunov function and often the judicious trial-and-error approach may be necessary. However, the use of some general properties of mechanical systems often gives good results. For instance, tools based on energy balance and the first integrals are usually very effective, see e.g. the monograph Glendinning (1994). If a system has several first integrals, the Lyapunov function can be written in a form of a linear combination of first integrals and possibly of their functions. The coefficients of these linear combinations, which can be considered as Lagrange multipliers, must be so determined as to get to the resulting function the properties of positive definiteness.

3. INSPECTION OF STABILITY OF RAYLEIGH-DUFFING EQUATION

Let us concentrate on the case when the motion is allowed in rotation only. Though simplified, this assumption allows us to analyze some important properties and to understand the role of individual parameters in the system. Investigation of the stability of nonlinear system around stationary points can be started by means of the corresponding linear system. However, for example, no conclusion can be drawn when the critical point is a center of the corresponding linear system. Also, for an asymptotically stable critical point it may be important to investigate the domain of asymptotic stability.

The system can be written in the system of first order equations:

$$\begin{aligned} \dot{\varphi} &= \psi \\ \dot{\psi} &= -\left[2\omega_{b\varphi} - K_J b_{\varphi\varphi}\left(1 - \gamma_{\varphi\varphi}\psi^2\right)\right]\psi - \\ &\quad - \left[\omega_\varphi^2 - K_J c_{\varphi\varphi}\left(1 - \beta_{\varphi\varphi}\varphi^2\right)\right]\varphi \end{aligned} \tag{3}$$

The position of the critical points can be obtained solving the respective algebraic systems which follows from Eq. (3) demanding, that the first derivatives on the left-hand side of equations vanish. Each of these systems admits three independent solutions. The first one corresponds to the case, when both the velocity and the rotation vanish providing the trivial solution:

$$\mathrm{P}_1: \quad \varphi = 0\,; \quad \psi = 0 \tag{4}$$

The second and the third solution implies that the velocity vanish as it follows from the first equation, while the second equation leads to the formula:

$$\mathrm{P}_{2,3}: \varphi = 0\,; \psi = \pm\sqrt{\frac{(K_J c_{\varphi\varphi} - \omega_\varphi^2)}{(K_J \beta_{\varphi\varphi} c_{\varphi\varphi})}} = \pm a \tag{5}$$

The points P_2 and P_3 are symmetrical with respect to the origin, so that it is sufficient to analyze the stability of one of them only. In order to analyze the equilibrium stability in the point P_2, the transformation of the coordinates is useful. We use the transformation which shifts the origin to the point a.

$$\varphi = a + \xi\,; \quad \psi = 0 + \zeta \tag{6}$$

For the system of transformed Rayleigh-Duffing type of differential equation (3) then holds:

$$\begin{aligned} \dot{\xi} &= \zeta \\ \dot{\zeta} &= -\left[2\omega_{b\varphi} - K_J b_{\varphi\varphi}(1 - \gamma_{\varphi\varphi}\zeta^2)\right]\zeta - \\ &\quad - \left[\omega_\varphi^2 - K_J c_{\varphi\varphi}(1 - \beta_{\varphi\varphi}(u + a)^2)\right](\xi + a) \end{aligned} \tag{7}$$

The linear variational equation of the vector field given by the right side of the Eqs (3) leads to the characteristic equation. Characteristic polynomial for the equilibrium point P_1 is given by following formula:

$$\Delta = \lambda^2 + \lambda(2\omega_{b\varphi} - b_{\varphi\varphi}K_J) + \omega_\varphi^2 - c_{\varphi\varphi}K_J \tag{8}$$

In the point P_2 the characteristic polynomial is given by:

$$\Delta = \lambda^2 + \lambda(2\omega_{b\varphi} - b_{\varphi\varphi}K_J) - 2\omega_\varphi^2 + 2c_{\varphi\varphi}K_J \tag{9}$$

Eigenvalues in the equilibrium points can be now easily calculated. They are necessary in the evaluation of the stability of the non-linear original system except some (critical) cases. In P_1 they can be calculated from the Eq. (8) and are given by the following formula:

$$\begin{aligned}\lambda_{1,2} = & -\tfrac{1}{2}(2\omega_{b\varphi} - b_{\varphi\varphi}K_J)\pm \\ & \pm\tfrac{1}{2}\sqrt{(2\omega_{b\varphi} - b_{\varphi\varphi}K_J)^2 - 4(\omega_\varphi^2 - c_{\varphi\varphi}K_J)}\end{aligned} \tag{10}$$

For the eigenvalues in the point P_2 a similar formula can be derived using Eq. (9).

The stability analysis comes out from the Jacobi matrix. We plot some particular examples to see, how the course of the Lyapunov function derivative changes on the trajectory, representing the solution of the system and therefore also to acknowledge, whether the stationary points are stable or not. Certainly, this phenomenon depends on the selection of parameters. At the Figs 2 and 3, the Lyapunov functions Φ and their derivatives Ψ are represented by the contour lines. The dashed lines stand for negative values, on the bold lines the functions vanish and on the continuous lines the function values are positive.

Having in hand the formulas for eigenvalues, we observe that two parameters are important in order to judge the stability of the stationary points. In the case of Rayleigh-Duffing Eq. (3) they are given by the formulas:

$$r_{1R} = 2\omega_{b\varphi} - K_J b_{\varphi\varphi}\,, r_{2R} = \omega_\varphi^2 - c_{\varphi\varphi}K_J \tag{11}$$

The Fig. 2 demonstrates the example of Lyapunov function and its time derivative for the system described by the Eq. (3) in the vicinity of the point P_1. On the Fig. 3 the example of functions in the vicinity of the point P_2 is demonstrated. The formulae for them are given as follows:

$$\begin{aligned}\Phi &= \varphi^2 + \psi\varphi + \psi^2/2 \\ \Psi &= -(\varphi+\psi)\left(3\varphi^3 + \psi\left(6\psi^2+1\right)\right) - \varphi^2\end{aligned} \tag{12}$$

The function Φ is positive everywhere around the origin, while the function Ψ is negative.

Let us investigate the stability and bifurcation of a stationary points of the system (3) in the case, where the eigenvalues of the relevant Jacobi matrix makes a pure imaginary couple. This case occurs, if the following conditions are fulfilled:

$$r_1 = 0 \quad ; \quad r_2 > 0 \tag{13}$$

If the eigenvalues of the Jacobi matrix cross transversally the imaginary axis, the change of stability character occurs. This phenomenon is demonstrated below on Figs 4 and 5. Moreover,

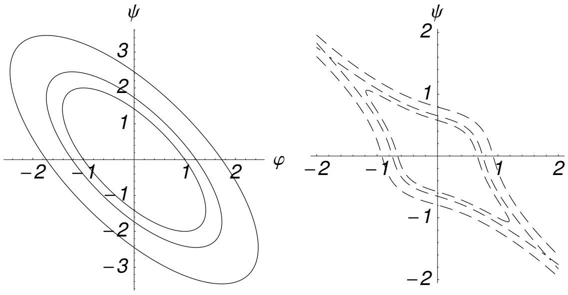

Figure 2: *An example of the Lyapunov function (left) and its derivative (right) for the system (3) in the neighborhood of the point* P_1. $r_{1R} > 0, r_{2R} > 0$ *and* $r_{1R} - 4\,r_{2R} > 0$. *The equilibrium point is stable.*

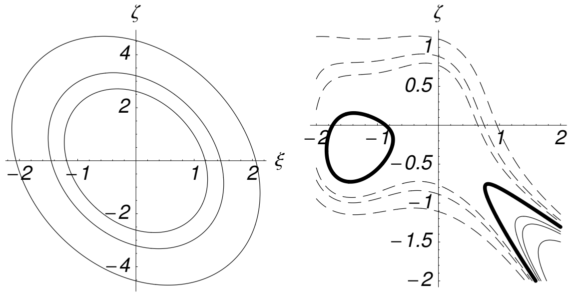

Figure 3: *An example of the Lyapunov function (left) and its derivative (right) for the system (3) in the neighborhood of the point* P_2. $r_{1R} > 0$, $r_{2R} < 0$ *and* $r_{1R} - 8\,r_{2R} > 0$. *The equilibrium point is stable.*

we obtain the "degenerated" case of Eq. (3) written as follows:

$$\begin{aligned}\dot\varphi &= \psi \\ \dot\psi &= -b_{\varphi\varphi}K_J\gamma_{\varphi\varphi}\psi^3 - \\ &\quad -(\omega_\varphi^2 - K_J c_{\varphi\varphi})\varphi - \beta_{\varphi\varphi}K_J c_{\varphi\varphi}\varphi^3\end{aligned} \tag{14}$$

The graph on the left of the Fig 4 shows an example of the solution of Eq. (14). By means of transforming the system into normal form and applying Lyapunov function and invariance principle we observe, that the origin is asymptotically stable and we treat the case of the pure imaginary eigenvalues and spiral point. A special case arises, when both $r_1 = 0$ and $r_2 = 0$, which is shown on the left graph of the figure. The origin is still asymptotically stable, but the approach to the equilibrium is not exponential, because of the hyperbolicity of the Eq. (14). The right graph represents the case, when the second condition (13) is violated. It is the case of pitchfork bifuraction when $r_1 = 0$ and $r_2 < 0$. Eigenvalues are real with identical absolute value. The origin

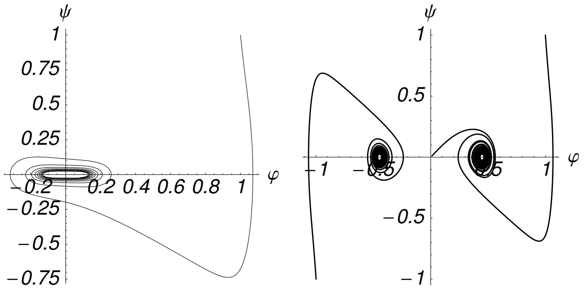

Figure 4: *Phase diagrams for the equation of motion, given by the Eq. (14)*

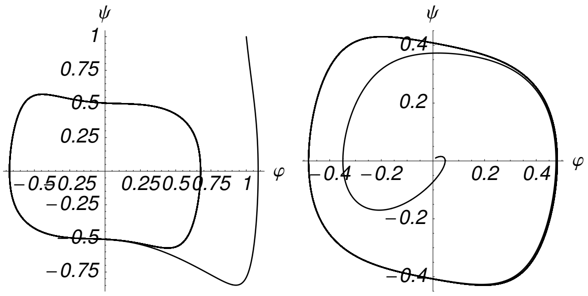

Figure 5: *The oscillation with occurrence of limit cycle.*

P_1 is unstable and there are two other asymptotically stable equilibrium points P_2, P_3.

On the Fig. 5 the supercritical limit cycle birth is shown. Passing with eigenvalues through the imaginary axis transversally, we get instable origin and a limit cycle. The left graph represents the case, when the origin is asymptotically stable respecting the conditions $r_1 > 0$ and $r_2 > 0$. The right diagram demonstrates the behaviour in the case of conditions $r_1 > 0$ and $r_2 > 0$.

4. NUMERICAL AND EXPERIMENTAL SOLUTION

Experimental activities in aeroelasticity have been focused for longer time on gaining of the knowledge of so-called aeroelastic derivatives and on the determining of the critical state. This is nowadays the well known task, which however does not comply with the real flutter nature, where the loss of stability is "sudden" and with large amplitude motion. From the point of view of basic research as well as from the perspective of the structural serviceability and lifetime, it is interesting therefore to get information predominantly regarding the system behaviour not only before the crisis appears, but also during the transition time as well as in the post-critical state.

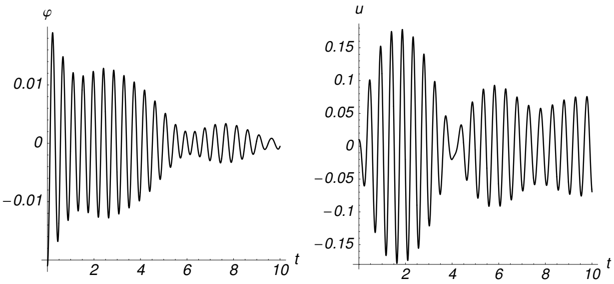

Figure 6: *Time history of the rotation φ (left) and the displacement u (right) taken from the numerical calculation of the system. The coefficients β_{ij}, γ_{ij} etc.*

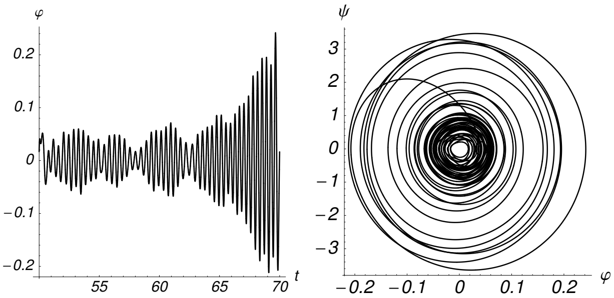

Figure 7: *Response and phase diagram from experiments with increasing velocity. Due to the influence of non-linear terms, the oscillation has several bumps before the limit cycle oscillation starts.*

Several numerical solutions and experiments have been done in order to investigate the system behaviour for varying equation parameters and initial conditions. The time history of the response is very sensitive to even slight change of them. An example of experimental results is given on the Fig. 6. Around critical points, the response amplitudes rise dramatically, stabilize and rise again and energy transmission between two modes is identified. It has been calculated for certain set of parameters, including the wind speed.

Using experimental rig described in Náprstek et al. (2007) the forces and amplitudes on several section models have been measured, not only during the low amplitude vibration, but also during and after the instability onset. One such sample of time history of the limit cycle oscillation birth is shown on the Fig. 7. The wind speed has been increasing continuously during the wind tunnel experiments. The pitch motion of the deck was small until it reached the bifurcation point. Keeping the wind speed constant, it reached the limit cycle apparently of Rayleigh-Duffing type. Three shapes of the deck cross-section were studied. The amplitudes of both the heave and pitch motions were registered using accelerometers. Pressures from twelve sensors located on a surface of the deck was recorded during experiments. This strategy enabled to compare not only the response of the deck but also the pressures in the individual points on the surface.

5. CONCLUSIONS

The non-linear mathematical model of the bending-torsional flutter has been composed and verified considerably. Duffing and Rayleigh non-linear terms in multi-component form with cyclic symmetry have been introduced into the differential system. They act as consistent expansion of the linear approach. The process of the mathematical model composition followed rather intuitive steps and trials to present an inherent description of effects and processes known from experimental measurements and numerical 2D/3D simulations.

The basic properties of the non-linear system have been investigated when passing into the non-stable domain. Possibilities of the post-critical re-stabilisation on the level of stable limit cycles (if any) are discussed. The widely known effects typical for post-critical regimes of various types can be described by means of the proposed non-linear model. In particular pitchfork bifurcation points have been detected with obvious configuration of stable and unstable branches; conditions of existence and relevant portraits of the principal limit cycles have been carried out; energy transflux between individual degrees of freedom has been detected and quantified. The development of these methods is in progress and relevant results will be provided in the near future.

After the loss of stability of trivial solution the response in one degree-of freedom tends to stabilize itself in the form of approximate harmonic or polyharmonic solution. The growth of this component after the stability loss of the trivial solution is prevented by a significant energy barrier. The system tends to the response interacting with the other component. When this energy barrier has been overcome, however, the system loses stability and its response grows beyond all limits. This typical case has a number of intermediate steps and special states, the origin of which depends on the appropriate combination of parameters described in the text.

ACKNOWLEDGMENT

The support of the Grant Agency of the Czech Republic under Grants No. 103/07/J060, No. 103/06/0099, No. A2071401 as well as the institutional support AV 0Z 20710524 is acknowledged.

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