High-order RKDG methods for computational electromagnetics

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Abstract

We introduce a new Runge–Kutta discontinuous Galerkin (RKDG) method for problems of wave propagation that achieves full high-order convergence in time and space. For the time integration it uses an mth-order, m-stage, low storage strong stability preserving Runge–Kutta (SSP–RK) scheme which is an extension to a class of non-autonomous linear systems of a recently designed method for autonomous linear systems. This extension allows for a high-order accurate treatment of the inhomogeneous, time-dependent terms that enter the semi-discrete problem on account of the physical boundary conditions. Thus, if polynomials of degree k are used in the space discretization, the (RKDG) method is of overall order $m = k + 1$, for any $k > 0$. Numerical results in two space dimensions are presented that confirm the predicted convergence properties.

Keywords: Discontinuous Galerkin methods; Wave propagation; Maxwell equations

1. Introduction

In this paper, we devise a new Runge–Kutta discontinuous Galerkin (RKDG) method that achieves full high-order convergence in time and space while keeping the time-step proportional to the spatial mesh-size. To this end, we derive an extension to non-autonomous linear systems of the mth-order, m-stage strong stability preserving Runge–Kutta (SSP-RK) scheme with low storage described in Gottlieb et al. [1]. With this time-integration scheme, and if polynomials of degree k are used in the space discretization, our RKDG method can be made to converge with overall order $m = k + 1$, for any $k > 0$. In particular, the scheme allows for a high-order accurate treatment of the inhomogeneous (time-dependent) terms that enter the semi-discrete problem on account of the physical boundary conditions.

2. The problem and the main result

2.1 Maxwell's equations

Consider the Maxwell's equations

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \quad \frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H} \tag{1}$$

where the magnetic induction **B** and electric displacement **D** will be assumed to depend linearly on the magnetic field **H** and **E** respectively, that is $\mathbf{B} = \mu_r \mathbf{H}$, $\mathbf{D} = \epsilon_r \mathbf{E}$. The Eqs. (1) must be supplemented with initial conditions

$$\mathbf{E}(x,0) = \mathbf{E}_0(x), \quad \mathbf{H}(x,0) = \mathbf{H}_0(x) \tag{2}$$

and boundary conditions that guarantee the continuity of tangential components of the electromagnetic field across material interfaces Γ

$$\mathbf{E}^+(x,t) \times \mathbf{n} = \mathbf{E}^-(x,t) \times \mathbf{n}, \ \mathbf{H}^+(x,t) \times \mathbf{n} = \mathbf{H}^-(x,t) \times \mathbf{n}, \quad x \in \Gamma, \tag{3}$$

where **n** is normal to Γ,

$$\mathbf{E}^+(x,t) \equiv \lim_{\tau \to 0^+} \mathbf{E}(x + \tau \mathbf{n}, t), \quad \mathbf{E}^-(x,t) \equiv \lim_{\tau \to 0^-} \mathbf{E}(x + \tau \mathbf{n}, t)$$

and similarly for $\mathbf{H}^{\pm}$. Finally, in the case of (exterior) *scattering* problems, an additional condition must be imposed at infinity. The physically relevant condition is the *Silver-Müller radiation condition* which requires that

$$\lim_{|x| \to \infty} \left[(\mathbf{H} - \mathbf{H}^{\text{inc}}) \times x - |x| (\mathbf{E} - \mathbf{E}^{\text{inc}}) \right] = 0 \tag{4}$$

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Computational Fluid and Solid Mechanics 2005
K.J. Bathe (Editor)

where ($\mathbf{E}^{inc}$, $\mathbf{H}^{inc}$) denotes the incident field.

2.2. Space discretization: a DG scheme

The Eqs. (1) can be written in 'conservative form' as

$$\mathbf{Q}\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{U}) = 0 \tag{5}$$

where

$$\mathbf{Q}(x) = \begin{bmatrix} \mu & 0 \\ 0 & \epsilon \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} \mathbf{H} \\ \mathbf{E} \end{bmatrix}, \quad \mathbf{F}_i(\mathbf{U}) = \begin{bmatrix} e_i \times \mathbf{E} \\ -e_i \times \mathbf{H} \end{bmatrix}$$

and $\mathbf{F}(\mathbf{q}) = [\mathbf{F}_1(\mathbf{q}), \mathbf{F}_2(\mathbf{q}), \mathbf{F}_3(\mathbf{q})]^T$. Given a partition $T_h = \{K_n\}$ of a computational domain Ω we consider local spaces $P^k(K_n)$ on each sub-domain consisting of polynomials of degree smaller than or equal to k. A DG formulation then takes on the form

$$\int_{K_n} \mathbf{Q}\frac{\partial U_h}{\partial t}\psi dx - \int_{K_n} F_h \cdot \nabla \psi dx + \int_{\partial K_n} \widehat{F_n} \cdot \mathbf{n}\psi ds = 0 \tag{6}$$

for all $\psi \in P^k(K_n)$, where $\mathbf{n}$ is the outward unit normal vector to ∂K_n and $\widehat{F_h} \cdot \mathbf{n}$ is the *numerical flux* on ∂K_h. A natural choice of the numerical fluxes is the upwind flux. For the present case, the upwind flux can be written as

$$\widehat{F} \cdot \mathbf{n} = \begin{bmatrix} \mathbf{n} \times \hat{\mathbf{E}} \\ -\mathbf{n} \times \hat{\mathbf{H}} \end{bmatrix} = \begin{bmatrix} \mathbf{n} \times \frac{(Y\mathbf{E} - \mathbf{n}\times\mathbf{H})^- + (Y\mathbf{E} + \mathbf{n}\times\mathbf{H})^+}{Y^- + Y^+} \\ -\mathbf{n} \times \frac{(Z\mathbf{H} + \mathbf{n}\times\mathbf{E})^- + (Z\mathbf{H} - \mathbf{n}\times\mathbf{E})^+}{Z^- + Z^+} \end{bmatrix} \tag{7}$$

where $Z = \sqrt{\mu/\epsilon}$ denotes the impedance and $Y = 1/Z$; see Mohammadian et al. [2].

For exterior (e.g. scattering) problems, a suitable approximation of the radiation condition, Eq. (4), must be imposed on an artificial boundary that truncates the computational domain and allows for reflected waves to exit with minimal reflections.

The exact non-reflecting conditions in Grote and Keller [3] can be used to derive (rather complicated) expressions for the fluxes at the artificial boundary. They give rise to *time-dependent terms* in the system of (ordinary differential) equations – cf. Eq. (9) – for the coefficients $\mathbf{c}_j^K(t)$ in the expansion

$$U_h|_K = \sum_{j=1}^{N_k} \mathbf{c}_j^K(t)\,\psi_j(x) \tag{8}$$

of the approximate fields in terms of the basis functions $\psi_j \in P^k(K)$.

2.3. Time discretization: an SSP-RK method for inhomogeneous systems

The Eq. (6), together with the Eq. (8) and numerical fluxes as described above, leads to a system of equations

$$\frac{d}{dt}\mathbf{C} = \mathbf{LC} + S(t) \tag{9}$$

where $\mathbf{L}$ is a constant matrix and $\mathbf{C}(t) = \left(\mathbf{c}_j^K(t)\right)_{\substack{1\le j\le N_k \\ K\in T_h}}$. To approximate the solution to Eq. (9), we propose a scheme which is an extension to the SSP-RK scheme introduced in Gottlieb et al. [1] for autonomous systems. Denoting $\mathbf{C}^n \equiv \mathbf{C}(t_n)$, we seek an mth order, m-stage scheme in the form

$$\mathbf{C}^{(0)} = \mathbf{C}^n$$

$$\mathbf{C}^{(i)} = \mathbf{C}^{(i-1)} + \Delta t L\mathbf{C}^{(i-1)} + \Delta t S^{(i)}, \quad i = 1, \ldots, m$$

$$\mathbf{C}^{n+1} = \sum_{k=0}^{m} \alpha_{m,k}\mathbf{C}^k \tag{10}$$

where the coefficients $\alpha_{m,k}$ are those corresponding to the scheme in Gottlieb et al. [1], namely

$$\alpha_{1,0} = 1$$

$$\alpha_{m,k} = \frac{1}{k}\alpha_{m-1,k-1}, \quad k = 1, \ldots, m-2$$

$$\alpha_{m,m} = \frac{1}{m!}, \quad \alpha_{m,m-1} = 0, \quad \alpha_{m,0} = 1 - \sum_{k=1}^{m}\alpha_{m,k} \tag{11}$$

and the source terms S^i are derived in Chen et al. [4]

$$S^{(i)} = (Id + \Delta t\partial_t)^{i-1}\phi(t_n) \tag{12}$$

2.4. Numerical results

We consider a radiation problem:

$$E_x(x,y,t) = \frac{y}{r}(-\sin(\omega t)J_1(\omega r) + \cos(\omega t)Y_1(\omega r))$$

$$E_y(x,y,t) = -\frac{x}{r}(-\sin(\omega t)J_1(\omega r) + \cos(\omega t)Y_1(\omega r))$$

$$H_z(x,y,t) = \cos(\omega t)J_0(\omega r) + \sin(\omega t)Y_0(\omega r) \tag{13}$$

where $r = \sqrt{x^2 + y^2}$ and $\omega = 13$. The computational domain Ω, Fig. 1, is a region with its outer and inner boundary being circles with radii $R = 1$ and $R = 3$, respectively. To test the accuracy of the algorithm in the absence of additional approximation due to the treatment of absorbing boundary condition, we impose the exact time-dependent boundary conditions on the boundary of the computation domain. We present results corresponding to the use of $P4$-elements and the SSP-RK 5 scheme. The relation between the error and

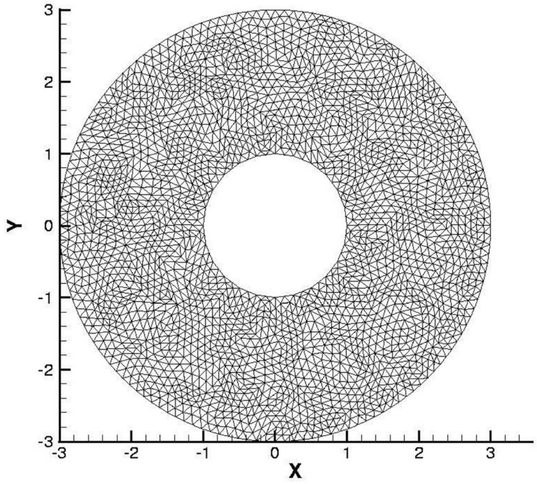

Fig. 1. Typical mesh for the test problem.

the number of elements (mesh size) is presented in Table 1, which displays the predicted order 5 convergence.

3. Conclusions

We have proposed a new RKDG method for computational electromagnetics that, for any given $k \geq 0$, attains $k + 1$ order of convergence in both space and time while maintaining the time-step proportional to the mesh size. A novelty of the method resides in the time-integration procedure for which it resorts to a suitable extension to non-autonomous systems of the mth-order, m-stage, low storage SSP-RK method introduced in Gottlieb et al. [1] for autonomous ordinary differential equations (ODEs). Numerical example confirmed the expected high-order rates of convergence. Finally, our basic considerations clearly extend to higher dimensions and to other linear, not necessarily hyperbolic, model equations. In particular, building on the present developments, forthcoming work on the Maxwell system will extend our high-order space/time implementations to full two-dimensional scattering configurations that include approximations to exact non-reflecting boundary conditions.

References

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Table 1
Convergence study: the errors are evaluated at $t = 0.05$

		P^4-elements and SSP-RK5			
N_e	h	L^1-error	Order	L^2-error	Order
72	1.1785e-01	1.3454e-02	–	4.4463e-03	–
290	5.8722e-02	2.3770e-04	5.7938	1.0076e-04	5.4365
1144	2.9566e-02	8.5100e-06	4.8525	3.8375e-06	4.7623
4556	1.4815e-02	2.8628e-07	4.9092	1.3125e-07	4.8853