On the treatment of uncertainties in structural mechanics and analysis

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Abstract

In this paper the need for a rational treatment of uncertainties in structural mechanics and analysis is reasoned. It is shown that the traditional deterministic conception can be easily extended by applying statistical and probabilistic concepts. The so-called Monte-Carlo simulation procedure is the key for those developments, as it allows the straightforward use of the currently used deterministic analysis procedures. Two numerical examples exemplify the methodology. It is concluded that uncertainty analysis may ensure robust predictions of variability, model verification, safety assessment, etc.

Keywords: Uncertainty; Monte-Carlo simulation; Finite elements; Response variability; Model verification

1. Introduction

Structural mechanics and analysis, up to this date, generally is still based on a deterministic conception [1]. Observed variations in loading conditions, material properties, geometry, etc., are taken into account by either selecting extremely high or low or average values, respectively, for representing the parameters. Hence, by this, uncertainties inherent in almost every analysis process are considered just intuitively. Observations and measurements of physical processes, however, clearly show their random characteristics. Statistical and probabilistic procedures provide a sound framework for a rational treatment of these uncertainties. Moreover there are various types of uncertainties to be dealt with (see Fig. 1). While the uncertainties in mechanical modeling can be reduced as additional knowledge becomes available, the physical or intrinsic uncertainties can not. Furthermore, the entire spectrum of uncertainties is also not known. In reality, neither the true model nor the model parameters are deterministically known. Assuming that by finite element (FE) procedures structures and continua can be represented reasonably well, the question of the effect of the discretization still remains. It is generally expected that an increase in the size of the structural models, in terms of degrees of freedom, will increase the level of realism of the model. Comparisons with measurements, however, clearly show that this expectation can not be confirmed. An ever-refined FE model just decreases the discretization error, but all other aspects contributing to the discrepancy between prediction and measurement will not be improved. There are several reasons for this. Among them is the fact that the FE model, which is a mathematical idealization, represents the physical behaviour not exactly but with a certain accuracy only. Typical examples for this are strongly non-linear interactions in a linear model, ignoring flexibilities at joints, inaccurate

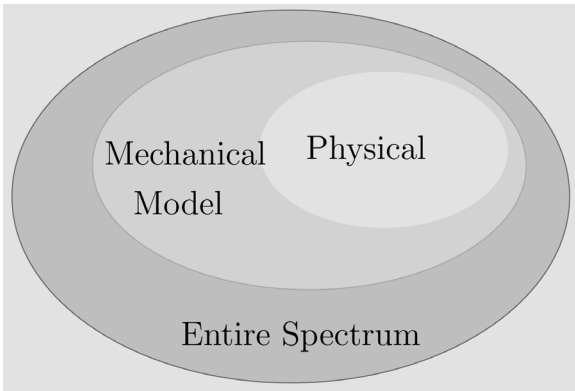

Fig. 1. Spectrum of uncertainties.

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modeling of the boundary conditions, ignoring the nonlinear interaction, etc. Furthermore, even if it is assumed that the idealized mathematical FE model represents the structural behavior, the model parameters do show uncertainties. As already stated above, these uncertainties refer to both loading – environmental loading such as water waves, wind, earthquakes, etc., are good examples of this – as well as to structural properties, such as imperfections of geometry, thickness, Young's modulus, material strength, fracture toughness, damping characteristics, etc. It is also well known that the results of experimental measurements are subjected to uncontrollable random effects. This is the main reason why they are so difficult to reproduce. This fact leads directly to the claim as made before, i.e. that an increase in the number of degrees of freedom does not compensate for the insufficient modeling of physical phenomena, such as not taking into account the uncertainties in the boundary conditions, etc. Needless to say that it is most important that the model reflects physical phenomena. This of course includes the uncertainties in both structural properties and loading conditions.

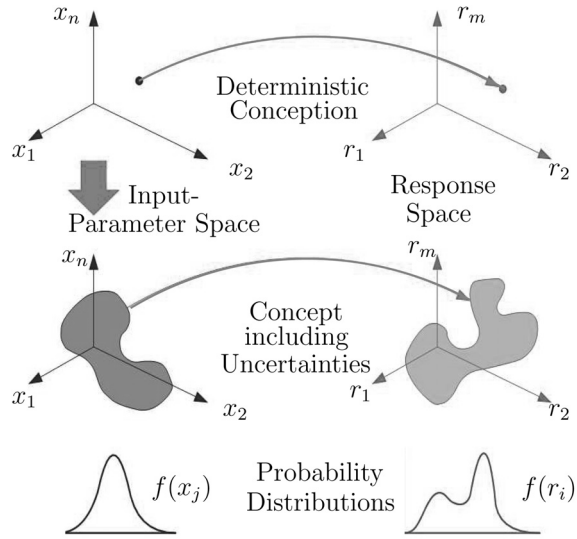

Fig. 2. Deterministic conception versus concept including uncertainties.

2. Modeling of uncertainties

While in the deterministic conception a single value is considered to suffice for the representation of a particular variable, there are in fact a great number of values – each associated with a certain probability of occurrence of a particular value – that are needed for a realistic description (Fig. 2). Hence the variables in their basic form may be described as so-called random variables X. Typical examples are, for example, the yield strength of materials etc. The associated uncertainty is quantified by probability measures, e.g. described as probability density functions. In other words, the probability that a parameter takes on values within an interval is

$$P(a < X \leq b) = \int_a^b f(x)\,\mathrm{d}x \tag{1}$$

This one-dimensional definition certainly can be expanded easily for multi-dimensional cases. The distribution of the occurrence of the various values, i.e. $f(x)$, also denoted as the probability density function, is generally characterized by certain types of function such as the normal or Gaussian distribution, etc. The parameters, such as the central tendency or mean value as well as the variance, are estimated by statistical procedures. For time variant processes, the probability density refers not only to one time instant, but to other times as well, i.e. to a family of random variables $X(t_1)$, $X(t_2)$ more simply denoted by $X(t)$. Again, if the distributions of $X(t_1)$, $X(t_2)$ are Gaussian, such a process is denoted as a Gaussian process [2]. Typical examples are wind, wave, earthquake records, etc. If the structural randomness includes spatially correlated random fluctuations of systems or load parameters as well (see, e.g. wind pressure fluctuations on area-like structures), the notion of a random field is used. A random field is generally defined by its type of probability distribution and the associated distribution parameters, such as mean and variance, the autocorrelation function, and other properties, such as homogeneity, etc. Homogeneity, for example, means that the statistical properties are independent from the specific location, which implies a constant mean value and a correlation function that depends only on the relative distance in space or time.

When specifying uncertainties, it is advisable to consider all uncertain parameters instead of considering just some 'most influential' uncertain parameters by a priori engineering judgment. This would prejudice the result, and high sensitivities of parameters not labeled as 'important' would not be reflected by the response. It is crucial to note that the computational effort for a single FE-run is basically independent of the number of random variables (uncertainties) introduced. Hence, considering a large number of uncertain parameters by using Monte-Carlo simulation, cf. Section 3, entails no disadvantage, while considering all uncertainties ensures a robust prediction of the variability. As a side effect, the variability of the response might also serve as a tool to verify, for example, the quality of the mathematical (FE-) model. A large scatter of the response reveals either an

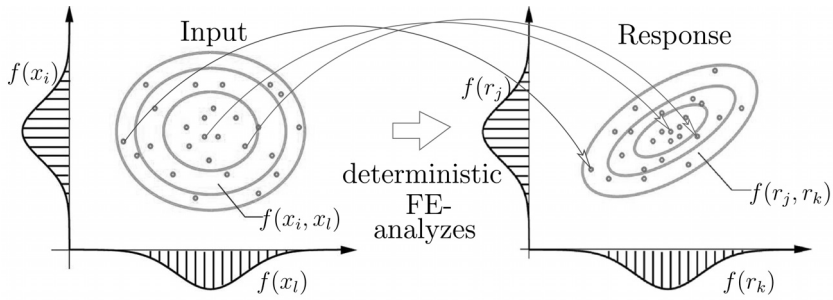

Fig. 3. Stochastic analysis based on Monte-Carlo sampling.

unfavorable high sensitivity on input parameters that are not sufficiently well known or indicates some modeling errors that require further improvements to arrive at robust predictions.

3. Methods and procedures

FE element models generally contain quite a large number of parameters like elasticity constants, geometry specifications, loading parameters, boundary conditions, etc., of which most values are not perfectly known [3]. It was stated above that the so-called 'true' parameters can, if at all, be determined in exceptional cases only, i.e. by experiments. Hence the values used in deterministic FE-analysis are so-called nominal values that deviate to a certain extent from the unknown true value. The uncertainties within the input parameters naturally result in uncertainties of the output, i.e. the response. Since response predictions are the central goal of any FE-analysis, and all predictions depend more or less on the uncertain input parameters, a rational approach has to include these unavoidable uncertainties.

Data are always scarce. Accessible statistical information, if any, might be restricted to the mean value, the standard deviation, upper and lower fractile values or upper and lower bounds. However, with this concept, whatever information is available can be used and – as new information or data becomes available – updated. Under these circumstances, it is reasonable to select the most convenient distribution that reflects the known or assumed variability (uncertainty) and avoids realizations that are not physically meaningful. For many reasons, the Gaussian normal distribution and log-normal distribution respectively are preferred. Since the uncertain input is specified mathematically by probability laws, the response follows also such laws, i.e. has a well defined unique distribution. Analytical methods to arrive at the distribution of the response require very specialized knowledge, and, most importantly, are not generally applicable, i.e. they are limited in their application and, in addition, not straightforward. The alternatives to analytical approaches are those based on what is denoted as Monte-Carlo simulation. This approach is most generally applicable, and all deterministic analysis tools can be integrated to their full extent.

Figure 3 depicts the basic principles of Monte-Carlo sampling [4] where the laws of statistics are exploited to derive information on the variability of the response. By using a suitable number generator (see e.g. RANDLIB [5]), statistically independent samples of the input are generated by a type of game of chance, and follow the prescribed probability distributions of the uncertain parameters. Suppose that each of the introduced random variables is represented by a component X_j of the vector $\mathbf{X} = \{X_j\}_{j=1}^{n}$. Hence, the input distribution $f(x_1, x_2, \ldots, x_n)$ is represented according to statistical laws by a finite number N of independent samples $\{\mathbf{x}^{(k)}\}_{k=1}^{N}$. Each vector $\boldsymbol{x}^{(k)}$ specifies for each uncertain parameter a deterministic discrete value and consequently defines deterministically the response that might be represented by the vector $\mathbf{r}^{(k)} = \mathbf{r}(\mathbf{x}^{(k)})$. Hence, traditional deterministic FE-analysis can be used to provide the mapping $\mathbf{r}^{(k)} = \mathbf{r}(\mathbf{x}^{(k)})$ between input and response.

In the simplest case, it might be justified to assume that all uncertainties are independent. Such as assumption is reasonable as long as this assumption does not contradict experience and physical properties. When the components are considered as independent, each component can be generated by available random number generators where the distribution and its parameters must be supplied. One can approximate the expectation or the mean of each response quantity r_i by

$$\mu_i = \mathrm{E}\{r_i\} \approx \frac{1}{\mathrm{N}} \sum_{\mathrm{k}=1}^{\mathrm{N}} r_i(\mathbf{x}^{(\mathrm{k})}) \tag{2}$$

and the variance by

$$\mathrm{Var}\{r_i\} = \sigma_i^2 = \mathrm{E}\{(r_i - \mu_i)^2\} \approx \frac{1}{\mathrm{N}-1} \sum_{\mathrm{k}=1}^{\mathrm{N}} (r_i(\mathbf{x}^{(\mathrm{k})}) - \mu_i)^2 \tag{3}$$

where $E\{\cdot\}$ denotes the expectation operator. In addition, the linear correlation between different response quantities can be computed:

$$\rho_{il} = E\{(r_i - \mu_i)(r_l - \mu_l)\}/(\sigma_i\sigma_j) \approx \frac{\sum_{k=1}^{N}(r_i(\mathbf{x}^{(k)} - \mu_i)(r_l(\mathbf{x}^{(k)} - \mu_l)}{\left[\sum_{k=1}^{N}(r_i(\mathbf{x}^{(k)} - \mu_i)^2 \cdot \sum_{k=1}^{N}(r_l(\mathbf{x}^{(k)} - \mu_l)^2\right]^{1/2}} \quad (4)$$

The above estimates (right-hand side expression) are random variables themselves, which fluctuate randomly. The fluctuation diminishes with an increasing sample size N, where the mean μ_i and standard deviation $\sigma_i = \sqrt{\mathrm{Var}\{r_i\}}$ have a coefficient of variation of $\approx \sqrt{1/N}$. For assessing the variability of the response, a sample size N in the order of 30 to 100 suffices for the first step to obtain estimates on the mean and the variance of the response. Hence the estimates will vary approximately 20% and 10% when using a sample size N of 25 and 100, respectively. This accuracy generally suffices to get a good idea of the variability of the response.

The above simple and straightforward procedure is getting more involved for cases where correlations between random variables need to be considered. Uncertain structural properties such as Young's modulus in a continuous beam or plate, the thickness of a plate, geometric imperfections of shells, fluctuation of pressure due to wind loading, and earthquake ground motion are continuous with respect to space or time. Hence, uncertain properties in the close neighborhood are generally strongly correlated, while the mutual dependency diminishes with the distance in space and/or time. Such continuous uncertain properties are described mathematically by random fields where the correlation coefficient tends to one as the distance tends to zero. When the uncertain properties are modeled by random variables, these variables, X_j, are now a function of time t (e.g. earthquake acceleration at a specific site), of the spatial position $\mathbf{p}$ (e.g. geometric imperfection of shells) or of both ($\mathbf{p}$, t), as in the case of wind pressure, which varies continuously with respect to time and space. The minimal information needed to describe a random field is the specification of the mean;

$$\mu_j(\mathbf{p}) = E\{X_j(\mathbf{p})\} \quad (5)$$

and the covariance,

$$C_j(\mathbf{p}_1, \mathbf{p}_2) = E\{(X_j(\mathbf{p}_1) - \mu_j(\mathbf{p}_1))(X_j(\mathbf{p}_2) - \mu_j(\mathbf{p}_2))\} \quad (6)$$

For the case where $f(x_j(\mathbf{p}))$ is normally distributed, the random field can be approximated by the so-called Karhunen–Loève expansion [6,7]:

$$X_j(\mathbf{p}) = x_j^{[0]}(\mathbf{p}) + \sum_{k=1}^{m} \xi_k \cdot x_j^{[k]}(\mathbf{p}) \quad (7)$$

where $\{x_j^{[k]}\}_{k=0}^{m}$ are deterministic functions with respect to time and space. This representation introduces just m independent standard normal random variables ξ_k with the property of zero mean and unit standard deviation:

$$E\{\xi_k\} = 0, \quad E\{\xi_k\xi_l\} = \delta_{kl} \quad (8)$$

with $\delta_{k=l} = 1$ and $\delta_{k\neq l} = 0$. The Karhunen–Loève expansion is related to the required specification of the mean and covariance by the following relations:

$$\mu_j(\mathbf{p}) = x_j^{[0]}(\mathbf{p}) \quad (9)$$

$$C_j(\mathbf{p}_1, \mathbf{p}_2) = \sum_{k=1}^{m} x_j^{[k]}(\mathbf{p}_1)x_j^{[k]}(\mathbf{p}_2) \quad (10)$$

For the case where no continuous random field is involved, but just discrete random variables that are correlated, a quite similar representation exists as shown above for random fields, where the free continuous vector $\mathbf{p}$ will be replaced by discrete values.

4. Practical applications

In order to demonstrate the applicability of the probabilistic approach described in the preceding sections, two examples are presented, in which uncertainties significantly affect the response predictions.

4.1. Coupled load analysis of a launcher-mounted satellite

In this first example, a large-scale FE-model of an aerospace application is analyzed, specifically the dynamic response of a satellite structure mounted on a launcher rocket. This is accomplished by reducing the problem with the sub-structuring technique known as 'Craig–Bampton' method [8]. Clearly, in a FE-model with this complexity and size, the uncertainty about its parameters is significant and unavoidable. In the present example, this random scatter is modeled in a probabilistic framework, by treating the parameters as random variables and estimating the resulting uncertainty in the response via Monte-Carlo simulation.

Figure 4 shows the FE-model of the launcher-satellite assemblage. It is divided into sub-structures, specifically the two lateral solid propellant boosters, the main stage and the upper composite at the top. The latter contains the payload, in the present case the satellite. The displayed sub-division reflects the decomposition of the numerical model applying the Craig–Bampton method. The entire model contains approximately 170 000 degrees of freedom, where the satellite contains roughly

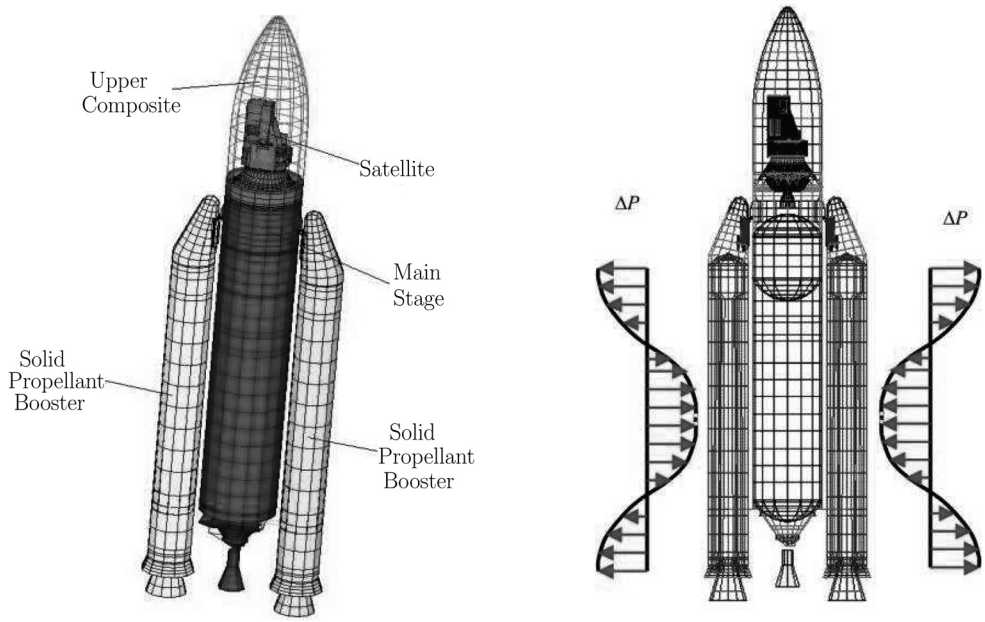

Fig. 4. (a) FE-model coupled launcher-satellite structure; (b) end-of-booster-pressure oscillation (EBP) load case.

120 000 DOFs, and the solid propellant boosters, the main stage and the upper composite approximately 12 000, 8 800 and 16 000 DOFs, respectively. The FE model of the satellite is significantly more refined than that of the launcher components, since the response quantities of interest are exclusively located in the satellite. After the Craig–Bampton reduction the number of DOFs of the coupled system amounts to approximately 800.

For the Monte-Carlo simulation, all material and geometry parameters specified in the model input files have been treated as random variables. This results in a total of around 1 300 random variables for both the satellite and the launcher. In order to separately assess the effects of the uncertainties in the launcher on one hand and in the satellite on the other, two sets of analyses have been performed. In the first set, the launcher properties have been varied randomly and the satellite set deterministically, and vice versa for the second set. The magnitude of the scatter in terms of the coefficient of variation is based on experimental data and on experience with similar structures and varies in the range of 4–6%. The distribution type is mostly Gaussian, with the exception of the viscous damping ratios, which are log-normal; for these the C.o.V. has been set significantly higher, namely 40%, since considerable uncertainty is usually associated with damping.

The load case considered in this example is the so-called end-of-booster-pressure oscillation (EBP), which takes place about two minutes after the ignition of the boosters at lift off. In essence, this load case simulates the pressure oscillation inside the combustion chamber of the solid propellant booster, cf. Fig. 4. The linear frequency response analysis covers the frequency range from 30 Hz to 54 Hz, with intervals of 0.2 Hz and has been performed with the commercial FE-code MSC.Nastran.

The purpose of the analysis is to estimate the effect of the uncertainties in the launcher and the satellite on the structural response. The considered critical response quantity is the von Mises stress in the beam connecting the solar panel to the satellite structure. The location of this beam is depicted in Fig. 5.

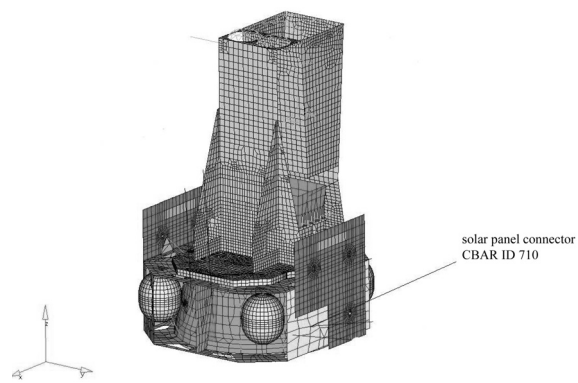

Fig. 5. Location of beam connecting the solar panel.

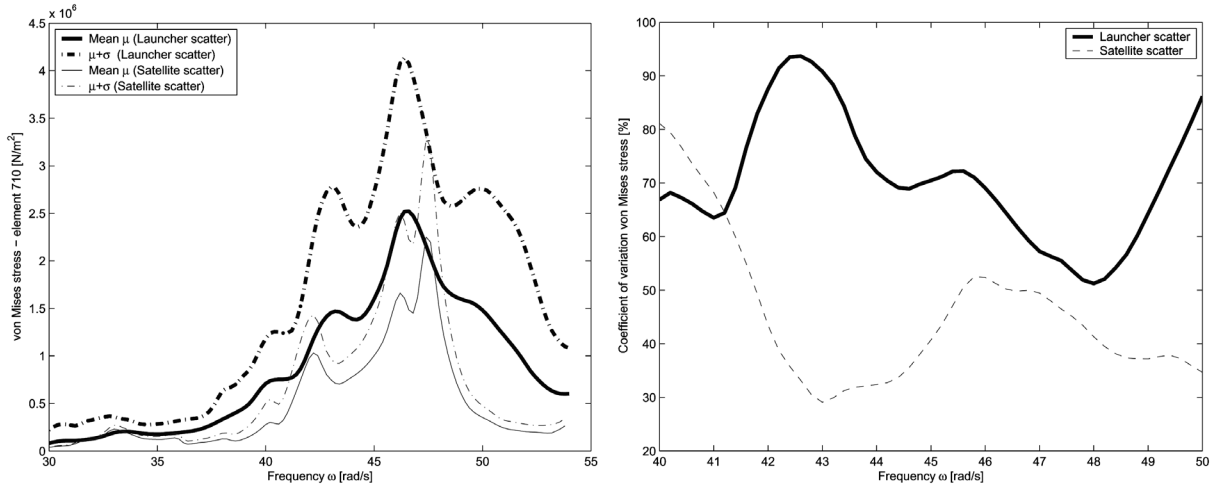

Fig. 6. (a) Frequency response of coupled launcher-satellite system, von Mises stress in solar panel connecting beam: (a) mean μ and mean shifted by standard deviation $\mu + \sigma$; (b) coefficient of variation σ/μ.

The results of the probabilistic analysis for the von Mises stress in the solar panel connector are synthesized in Fig. 6. The left portion of the figure shows the mean μ and the mean shifted by the standard deviation $\mu + \sigma$. The thick lines refer to the case in which the uncertainties in the launcher properties are captured, whereas the thin lines result when the satellite properties are modeled as random variables. In the right part of the figure, the coefficient of variation is shown. For this plot, the frequency range has been restricted to the interval [40,50] Hz, firstly because in this range the maximum responses occur and, secondly, because outside this range the mean values are very small and the coefficient of variation ceases to be a meaningful measure of the scatter.

Clearly one can see that the scatter and hence the uncertainty in the response is enormous. In Fig. 6 (a), the σ-shifted mean response exceeds the mean response by $\approx 60\%$ around the peak response of the analysis that captures the effect of the launcher scatter. In the range between 40 and 50 Hz the C.o.V. does not fall below 30% and, in the case of the launcher scatter analysis, even approaches 100%.

This shows that neglecting the uncertainty in the structural components and performing a single deterministic analysis would lead to fictitiously accurate results. A severe underestimation of the effects of the load case would most probably occur.

4.2. Thin-walled cylindrical shell with random geometric imperfections

As a second example, the effect of random geometric imperfections on the limit loads of isotropic, thin-walled, cylindrical shells under deterministic axial compression is presented [9]. Therefore, a concept for the numerical prediction of the large scatter in the limit load observed in experiments using direct Monte-Carlo simulation technique in connection with the finite element method is introduced. Geometric imperfections are modeled as a two-dimensional Gaussian stochastic field with prescribed second moment characteristics based on a database of measured imperfections [10]. In order to generate realizations of geometric imperfections, the estimated covariance kernel is decomposed into an orthogonal series in terms of eigenfunctions with corresponding uncorrelated Gaussian random variables, known as the Karhunen–Loève expansion (see Fig. 7). For the determination of the limit load a geometrically nonlinear static analysis [11] can be carried out using a general purpose code, e.g. STAGS [12], where one typical result is shown in Fig. 8.

The second moment characteristics of the limit load are obtained by using the direct Monte-Carlo simulation. In fact, the numerically predicted statistics of the limit loads coincide reasonably well with the actual observations as shown in Fig. 9. It might be observed, though, that the calculated mean value is somewhat higher when compared with the experimentally determined mean. This is due to the fact that there are still a number of uncertainties – in addition to the geometric imperfections – that have not yet been considered in this analysis, e.g. varying thickness and material properties (Young's modulus), imperfect boundary conditions, misalignment in loading, etc.

Fig. 7 Geometric imperfections using the Karhunen–Loève expansion.

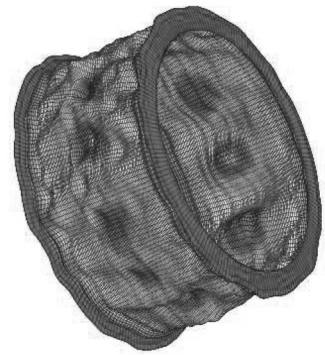

Fig. 8. Deformation at the limit load for the shell by STAGS [11].

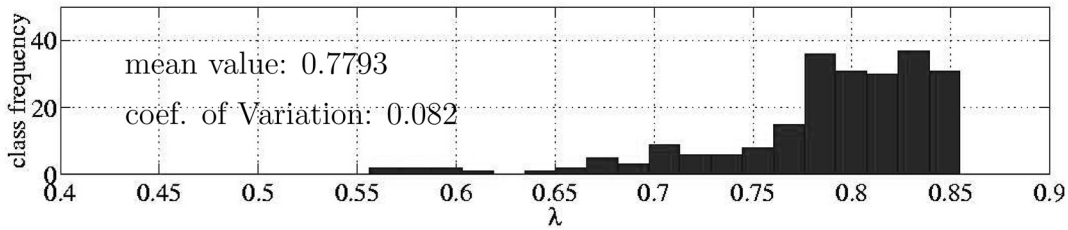

Fig. 9. For comparison: experimental observations [9]: mean value $\mu = 0.6433$, coefficient of variation $V = 0.0867$.

5. Summary and conclusions

In the pre-computer times, and during its early stages of development, the computational efforts to process uncertainties were prohibitive. In those days the use of the deterministic conception, which in fact is a simplification of the realistic situation, was absolutely necessary. But the breathtaking developments in computer technology (hardware) have laid the ground for the reconsideration of the traditional procedures. The possibility of parallel and distributed computing, the further development of Monte-Carlo simulation procedures, etc., on the software side accelerate this process even more. It is envisaged that before long, uncertainty analysis will be an inherent part of engineering structural analysis. Such uncertainty analysis ensures the robust prediction of variability, among many other possibilities, such as model verification, safety assessment, etc.

References

[1] Bathe K-J. Finite Element Procedures. Englewood Cliffs; NJ: Prentice Hall, 1996.
[2] Y.K. Lin. Probabilistic Theory of Structural Dynamics. New York: Krieger, 1976.
[3] Schuëller GI, Pradlwarter HJ, Schenk CA. Nonstationary response of large linear FE-models under stochastic loading. *Comput Struct* 2003;81(8–11):937–947.
[4] Fishman GS. Monte Carlo: Concepts, Algorithms, and Applications. New York: Springer 1996.
[5] Brown BW, Lovato J, Russel K, Venier J. RANDLIB Library of Routines for Random Number Generation (C and Fortran) http://utmdacc.mdacc.tmc.edu/
[6] Loève M. Probability Theory, 3rd edn. Princeton, NJ: Van Nostrand, 1963.
[7] Ghanem RG, Spanos PD. Stochastic Finite Elements: A Spectral Approach. New York: Springer, 1991.
[8] Craig RR, Bampton MCC. Coupling of substructures for dynamic analyses. AIAA Journal 1968;6(7):1313–1319.
[9] Schenk CA, Schuëller GI. Buckling analysis of cylindrical shells with random geometric imperfections. *Non-Linear Mechanics*, 2003;38:1119–1132.
[10] Arbocz J, Abramovich H. The initial imperfection data bank at the Delft University of Technology, part 1: technical report LR-290. Technical report, Delft University of Technology, Department of Aerospace Engineering, December, 1979.
[11] Riks E. Some computational aspects of the stability analysis of nonlinear structures. Comp Methods Appl Mech Eng 1984;47:219–259.
[12] Lockheed Martin Missiles and Space Co. Inc., Palo Alto, CA. STAGS (Structural Analysis of General Shells) User Manual, lmsc p032594, version 3.0, 1998.