Draw resonance revisited

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Abstract. We consider the problem of isothermal fiber spinning in a Newtonian fluid with no inertia. In particular, we focus on the effect of the downstream boundary condition. For prescribed velocity, it is well known that an instability known as draw resonance occurs at draw ratios in excess of about 20.2. We shall revisit this problem. Using the closed form solution of the differential equation, we shall show that an infinite family of eigenvalues exists and discuss its asymptotics. We also discuss other boundary conditions. If the force in the filament is prescribed, no eigenvalues exist, and the problem is stable at all draw ratios. If the area of the cross section is prescribed downstream, on the other hand, the problem is unstable at any draw ratio. Finally, we discuss the stability when the drawing speed is controlled in response to changes in cross section or force.

1. Introduction

Fiber spinning is a manufacturing process used in making textile or glass fibers. A highly viscous fluid is extruded vertically from a nozzle. It is then cooled by the ambient air and solidifies. The solidified fiber is then wound up on a spool at the end of the spinline.

Many physical effects are potentially significant in the study of this problem: viscosity, inertia, gravity, surface tension, cooling, elasticity and air drag may all be relevant. In this paper, we focus on the simplest model and study the influence of varying boundary conditions. We assume that the force in the fiber is purely due to viscous effects, and we ignore temperature dependence. We use a one-dimensional model based on slender geometry and cross sectional averaging. Let $u(x,t)$ denote the axial speed, and $A(x,t)$ the area of the cross section. The spinneret is located at $x = 0$ and the spool is at $x = L$. The conservation of mass implies that

$$A_t + (uA)_x = 0. \tag{1}$$

If only viscous forces contribute, the tension in the fiber is given by $3\eta A u_x$, where η is the viscosity. The requirement of constant tension in the fiber leads to

$$(Au_x)_x = 0. \tag{2}$$

Boundary conditions in industrial processes are notoriously ill-defined. It is customary to assume that A and u are given at the spinneret: $A(0,t) = A_0$, $u(0,t) = u_0$. This of course, is an idealization; in reality there is a transition to an upstream flow, which cannot be described by the one-dimensional model. At the spool, it is sensible to prescribe either the speed or the force with which the fiber is wound up. One might also consider control strategies where the flow is monitored and the speed of the spool adjusted to achieve a given objective. Since the goal of the manufacturing process is a fiber of uniform cross section, a control strategy might aim to keep the cross section constant. We shall consider what happens in the case of perfect success of such a control, i.e. when constant area is imposed as a boundary condition. We shall thus focus on the following three boundary conditions:

(i) Prescribed speed: $u(L,t) = u_1$.

(ii) Prescribed force: $A(L,t)u_x(L,t) = F$, where F denotes the force divided by the elongational viscosity 3η.

(iii) Prescribed cross section: $A(L,t) = A_1$.

It is easy to see that the problem admits the steady solution

$$u_s(x) = u_0 e^{kx}, \quad A_s(x) = A_0 e^{-kx}, \tag{3}$$

where, respectively,

$$e^{kL} = u_1/u_0, \quad k = F/(A_0 u_0), \quad e^{kL} = A_0/A_1 \tag{4}$$

for the three choices of boundary conditions. The dimensionless quantity $q = e^{kL}$ is called the draw ratio.

2. Linear stability

The stability of the steady solution was first analyzed by Kase et al. [2] and Pearson and Matovich [3]. For subsequent reviews and textbook chapters, see also [1, 4, 7, 8]. We note that much of the literature on draw resonance is concerned with the effect of additional physical mechanism, which are not included in our analysis. Inertia, elasticity and cooling generally have a stabilizing effect, while surface tension and shear thinning are destabilizing. This paper, on the other hand, will focus purely on the case of Stokes flow and investigate the effect of varying the downstream boundary condition. In the case of prescribed speed, an instability known as draw resonance is found for draw ratios in excess of about 20.2, while no such instability is found for prescribed force. In [6], the case of a linear combination of speed and force is also investigated; as expected, the stability threshold increases from 20.2 to infinity as the relevant coefficient is varied. The case of prescribed cross section does not seem to have been analyzed in the literature. We shall see that this boundary condition leads to instability at all draw ratios.

The reader is referred to [5] for the details of the linear stability analysis. It can be shown that the following characteristic equations are obtained (here μ is a normalized growth rate and q is the draw ratio):

For fixed speed:

$$(e^{\mu} - e^{\mu/q})q + (q - \mu)(\mathrm{Ei}(\mu) - \mathrm{Ei}(\mu/q)) = 0. \tag{5}$$

For fixed force:

$$e^{\mu} = 0. \tag{6}$$

For fixed cross section:

$$\mathrm{Ei}(\mu) - \mathrm{Ei}(\mu/q) = 0. \tag{7}$$

3. Behavior of the spectrum

In the case of fixed force, no eigenvalues exist, and it can indeed be shown in this case that initial disturbances will decay to zero in finite time [5]. For other boundary conditions, the asymptotic behavior of Ei for large argument can be exploited to approximate large eigenvalues. We refer to [5] for details. The stability is found to depend crucially on the boundary conditions. If the speed is described, then the flow becomes unstable beyond a critical draw ratio as is well known. If the area of the cross section is prescribed, the flow is unstable at all draw ratios. One might also consider control strategies where the speed is adjusted in response to changes in force or cross-section. While the former has the expected effect, the latter does not. Intuitively, one would be tempted to increase the drawing speed in response to larger cross-sectional area, but this turns out to be destabilizing. In addition, new instabilities emerge at low draw ratios.

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